Distributed optimal fusion filtering for singular systems with random transmission delays and packet dropout compensations

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ABSTRACT

This paper is concerned with the fusion filtering problem for time-varying singular systems with random transmission delays (RTDs) and packet dropout (PD) compensations. Here, the phenomena of RTDs and PDs are both characterized by Bernoulli distributed random variables with different probabilities. Generally, the current sensor measurement and one-step delayed sensor measurement can be received by filter. When the sensor measurement is lost, based on the strategy of PD compensations, the one-step predictor of current sensor measurement is used as compensator. Then, the new augmented systems with stochastic parameter matrices and correlated noises are introduced based on the measurement compensation model. Utilizing the innovation analysis approach, the local filters (LFs) dependent on probabilities and corresponding estimation error covariance matrices are derived for augmented systems. Moreover, the matrix-weighted distributed fusion filter (DFF) is designed for original singular systems on the basis of the state transformation. Compared with the LFs, it is not difficult to see that the presented DFF has better precision. In the end, some comparison simulation experiments are carried out to verify the effectiveness of the proposed fusion filtering algorithm.

1. Introduction

The time-varying singular systems (TVSSs) are the class of dynamics systems, which have more widespread applications than the normal time-invariant systems [1]. For instance, the mathematical models are used to describe the TVSSs with applications in chemical, physical, ecological, economic and biological domains. However, compared with the normal systems, the TVSSs cannot guarantee the structural stability owing to the perturbations of system parameters. Due to the fact that the structure of singular systems is complex as well as novel, it is difficult to investigate the state estimation (SE)

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problem for singular systems [2]. Accordingly, the issue of the SE for singular systems has long been one of the mainstream topics [3,4]. In recent years, owing to the development of the hardware technology, the multi-sensor systems have been extensively used, such as in the intelligent robot systems and unmanned vehicles [5–9]. Accordingly, the multi-sensor information fusion estimation (MSIFE) problem has attracted increasing attention [10–12]. With regard to the multi-sensor singular systems (MSSSs), there exist different methods to handle MSIFE problem, which include the reduced-order estimation, the full-order estimation and so on. For instance, the full-order fusion filter weighted by matrices has been designed in [13] for multi-delay stochastic singular systems with multiple sensors. The weighted measurement fusion estimators have been presented in [14] for the reduced-order subsystems via the singular value decomposition method. Nevertheless, it is worthwhile to note that most of the published results are applicable for the time-invariant singular systems. In reality, almost all real-time systems have time-varying characteristics [15–18]. Consequently, the MSIFE problem for time-varying MSSSs deserves further investigation.

As we all know, the random transmission delays (RTDs) and packet dropouts (PDs) occur inevitably during data transmission through communication networks due to the limited communication bandwidth or network congestion [19–23]. Generally, the Bernoulli distributed random variables and Markovian chains can be utilized to describe the phenomena of RTDs and PDs [24,25]. To mention a few, the issue of the MSIFE has been investigated in [26] for a class of power systems in the presence of multi-rate measurements as well as randomly occurring measurement delays, where a diagonal matrix composed of a set of mutually independent random variables satisfying the Bernoulli distribution has been used to model the RTDs. Here, each diagonal element takes 0 or 1. When the value is 1, it means that the current measurement is received successfully by filter, otherwise, the filter can receive the previous measurement. In addition, the optimal filter has been presented in [27] for networked systems subject to the RTDs, where the process of delay has been described by a multi-state Markovian chain. On the other hand, when it comes to the phenomenon of PDs, the key issue is how to compensate the effects of the lost measurement. At present, many compensation schemes have been proposed to tackle the effects caused by PDs, such as the zero-input strategy [28–30], hold-input strategy [31,32] and prediction compensation strategy [33–36].

Recently, it is well recognized that the compensation schemes of the zero-input strategy and hold-input strategy are suboptimal in contrast with the prediction compensation strategy. This is because it only takes the latest measurement data into consideration and ignores the historical data received previously by filter. Moreover, the prediction compensation strategy attenuates significantly the impacts induced by the lost measurement data since, in case of loss, all the previous data received successfully are utilized to update the filter instead of using only the last one [37,38]. Nevertheless, most of the studies regarding the PD compensations have mainly focused on the normal systems. Unfortunately, with regard to the MSSSs, the issue of the MSIFE with PD compensation strategy is still open and remains challenging. In practical systems, the nonlinearity is one of the most ubiquitous phenomena which, if not sufficiently handled, would give rise to jeopardize the performance of systems [39,40]. In [41], the issue of the distributed fusion estimation has been discussed for a class of nonlinear uncertain systems with multiplicative random parameters and random measurement delays. In particular, the nonlinearities exist in almost all engineering systems. The nonlinearities can severely degrade the performance of systems if they cannot be handled properly. Recently, the stochastic nonlinearity as a special case including state-dependent multiplicative noise has also gained much attention [42–45]. Unfortunately, the existing results have not taken the RTDs and PD compensations into account for nonlinear singular systems, which consist of the main motivation by shortening such a gap.

Motivated by the above discussion, in this paper, we aim to develop the fusion filtering scheme for a class of TVSSs with multiple sensors subject to RTDs and PD compensations. The addressed problem is significant owing to the following substantial difficulties: (1) how to properly reflect the phenomena of RTDs and PDs with different probabilities; and (2) how to choose the compensator as the filter input when considering the occurrence of the PDs. Based on the aforementioned analysis, the main contributions of this paper are highlighted as follows: (1) the distributed fusion filtering issue is addressed, for the first time, for a class of nonlinear TVSSs with multiple sensors in the presence of RTDs and PD compensations; (2) the phenomena of RTDs and PDs are modeled by a set of Bernoulli distributed random variables with known probability; (3) the new augmented systems with stochastic parameter matrices are introduced to attenuate the complex calculations; and (4) the derived distributed fusion filter (DFF) via the projection theory and the matrix-weighted fusion estimation scheme, which has the feature of the robustness and flexibility due to the parallel structure, has better precision than the local filters (LFs).

Notation In this paper, I_k represents the identity matrix with dimension k. The superscript T represents the transpose of a matrix. $\mathrm{tr}(A)$ is used to describe the trace of matrix A. $\perp$ denotes orthogonality. $\mathrm{diag}\{A_1, A_2, \ldots, A_N\}$ stands for a diagonal matrix with diagonal element A_i $(i = 1, \ldots, N)$. $\left[A_{ij}\right]_{N\times N}$ represents the $N \times N$ matrix with element A_{ij} $(i, j = 1, \ldots, N)$. $\det(A)$ denotes the determinant of matrix A. $\mathrm{rank}(A)$ denotes the rank of matrix A. $\deg F(x)$ denotes the degree of polynomial $F(x)$. $\{*\}$ denotes the same item as the front neighboring term. δ_{sh} represents the Kronecker delta function.

2. Problem formulation

In this paper, we consider the following class of discrete time-varying stochastic systems with N sensors:

$$Mx_{s+1} = \Phi_s x_s + \Upsilon_s \varpi_s, \quad s \geq 0, \tag{1}$$

$$z_{i,s} = C_{i,s}x_s + f_i(x_s, \xi_{i,s}) + \vartheta_{i,s}, \quad s \geq 0, \quad i = 1, 2, \ldots, N, \tag{2}$$

where $x_s \in \mathbb{R}^n$ represents the state vector to be estimated, $z_{i,s} \in \mathbb{R}^{m_i}$ denotes the sensor measurement outputs, $\varpi_s \in \mathbb{R}^r$ is the zero-mean process noise, and $\vartheta_{i,s} \in \mathbb{R}^{m_i}$ is the zero-mean measurement noises. The subscript i denotes the ith sensor. The function $f_i(x_s, \xi_{i,s})$ is the stochastic nonlinearity, where $\xi_{i,s}$ is the zero mean Gaussian white noise sequence. M, Φ_s, Υ_s, and $C_{i,s}$ are known matrices with appropriate dimensions.

In order to prevent the network congestion, the measurement data from the sensor side is sent to the filter only once at each moment. One packet at most is employed to update the filter in time and avoid waiting, i.e., the first received packet will be used to update the filter. Furthermore, the current sensor measurement $z_{i,s}$, the one-step delay sensor measurement $z_{i,s-1}$ or nothing may be received by filter at each moment. The predictor $\hat{z}_{i,s|s-1}$ of the current sensor measurement $z_{i,s}$ is utilized to compensate for the impacts of packet loss when nothing is received. As a result, the new measurement model with prediction compensation is designed as follows:

$$\mathcal{Y}_{i,s} = \alpha_{i,s}z_{i,s} + (1-\alpha_{i,s})\left\{(1-\alpha_{i,s-1})\beta_{i,s}z_{i,s-1} + [1-(1-\alpha_{i,s-1})\beta_{i,s}]\hat{z}_{i,s|s-1}\right\}, \quad s \geq 1; \quad \mathcal{Y}_{i,0} = z_{i,0}, \tag{3}$$

where $\mathcal{Y}_{i,s} \in \mathbb{R}^{m_i}$ denotes the measurement received by the ith filter and $\hat{z}_{i,s|s-1}$ denotes the one-step predictor of the sensor measurement $z_{i,s}$. $\alpha_{i,s}$ and $\beta_{i,s}$ are Bernoulli distributed random variables with probabilities Prob $\{\alpha_{i,s} = 1\} = \bar{\alpha}_i$, Prob $\{\alpha_{i,s} = 0\} = 1 - \bar{\alpha}_i$, Prob $\{\beta_{i,s} = 1\} = \bar{\beta}_i$ and Prob $\{\beta_{i,s} = 0\} = 1 - \bar{\beta}_i$ satisfying $0 \leq \bar{\alpha}_i \leq 1$ and $0 \leq \bar{\beta}_i \leq 1$, which are uncorrelated with other random variables.

Assumption 1. M is a singular square matrix, i.e., $\text{rank}(M) = n_1 < n$.

Assumption 2. System (1) is regular, i.e., $\det(\kappa M - \Phi_s) \neq 0$, where κ is an arbitrary complex number.

Assumption 3. System (1) is causal, i.e., $\deg \det(\kappa M - \Phi_s) = \text{rank}(M)$, where κ is an arbitrary complex number.

Assumption 4. ϖ_s and $\vartheta_{i,s}$ have the following statistical properties:

$$\mathbb{E}\left\{\begin{bmatrix}\varpi_s \\ \vartheta_{i,s}\end{bmatrix}\begin{bmatrix}\varpi_h^T & \vartheta_{i,h}^T\end{bmatrix}\right\} = \begin{bmatrix}Q_{\varpi,s} & 0 \\ 0 & Q_{\vartheta,s}^i\end{bmatrix}\delta_{sh}, \quad \mathbb{E}\left\{\vartheta_{i,s}\vartheta_{j,h}^T\right\} = 0, \quad (i \neq j, \forall s, h).$$

Assumption 5. The initial state is x_0 with $\mathbb{E}\{x_0\} = \pi_0$ and $\mathbb{E}\left\{(x_0 - \pi_0)(x_0 - \pi_0)^T\right\} = P_0$, which is uncorrelated with ϖ_s, $\vartheta_{i,s}$, $\xi_{i,s}$, $\alpha_{i,s}$ and $\beta_{i,s}$.

Assumption 6. The function $f_i(x_s, \xi_{i,s})$ satisfies

$$\mathbb{E}\left\{f_i(x_s, \xi_{i,s})|x_s\right\} = 0, \quad \mathbb{E}\left\{f_i(x_s, \xi_{i,s})f_j^T(x_h, \xi_{j,h})|x_s\right\} = \sum_{l=1}^{m} \Pi_i^{(l)} x_s^T \Gamma_i^{(l)} x_s \delta_{sh}\delta_{ij}, \tag{4}$$

where m is a known positive integer, $\Pi_i^{(l)}$ and $\Gamma_i^{(l)}$ $(l = 1, 2, \ldots, m)$ are known positive semi-definite matrices with suitable dimensions.

Remark 1. It should be noted that the nonlinearities occur universally in a wide range of engineering practices. The addressed stochastic nonlinearity $f_i(x_s, \xi_{i,s})$ in (2) encompasses many well-studied nonlinear functions as a special cases. Generally, the nonlinear function $f_i(x_s, \xi_{i,s})$ could cover the following situations such as, the state-dependent multiplicative noises, the random vectors dependent on the norm of state and a random sequence dependent on the sign of a nonlinear function of state. As a special case involving state-dependent noise, the stochastic nonlinearity $f_i(x_s, \xi_{i,s})$ satisfying (4) represents the internal nonlinear disturbance and is independent of the sensor failures or network congestions.

Remark 2. The model (3) describes the phenomena of RTDs and PDs during data transmission through unreliable networks. We get $\mathcal{Y}_{i,s} = z_{i,s}$ when $\alpha_{i,s} = 1$, i.e., the current sensor measurement is received by filter on time. When $\alpha_{i,s} = 0$, $\alpha_{i,s-1} = 0$ and $\beta_{i,s} = 1$, we have $\mathcal{Y}_{i,s} = z_{i,s-1}$, i.e., the prior sensor measurement is received by filter at this time. When $\alpha_{i,s} = 0$, $\alpha_{i,s-1} = 1$ or $\alpha_{i,s} = 0$, $\beta_{i,s} = 0$, we have $\mathcal{Y}_{i,s} = \hat{z}_{i,s|s-1}$, i.e., the packet is lost at this time and then the predictor $\hat{z}_{i,s|s-1}$ is used as a compensator to update the filter. Therefore, we can obtain that the packet on time receiving probability is $\text{Prob}\left\{\alpha_{i,s} = 1\right\} = \bar{\alpha}_i$, the one-step delay receiving probability is $\text{Prob}\left\{\alpha_{i,s} = 0, \alpha_{i,s-1} = 0, \beta_{i,s} = 1\right\} = (1-\bar{\alpha}_i)^2\bar{\beta}_i$, and the compensation probability is $\text{Prob}\left\{\alpha_{i,s} = 0, (1-\alpha_{i,s-1})\beta_{i,s} = 0\right\} = (1-\bar{\alpha}_i)[1-(1-\bar{\alpha}_i)\bar{\beta}_i]$.

For systems (1) and (2), there are nonsingular matrices U and V as in [46] such that

$$UMV = \begin{bmatrix}I_{n_1} & 0 \\ 0 & 0\end{bmatrix}, \quad U\Phi_s V = \begin{bmatrix}\Phi_{11,s} & \Phi_{12,s} \\ \Phi_{21,s} & \Phi_{22,s}\end{bmatrix}, \quad U = \begin{bmatrix}U_1 \\ U_2\end{bmatrix},$$

$$U\Upsilon_s = \begin{bmatrix}\Upsilon_{1,s} \\ \Upsilon_{2,s}\end{bmatrix}, \quad C_{i,s}V = \begin{bmatrix}C_{i,s}^{(1)} & C_{i,s}^{(2)}\end{bmatrix}, \quad V^{-1}x_s = \begin{bmatrix}x_{1,s} \\ x_{2,s}\end{bmatrix},$$

where $\Phi_{22,s} \in \mathbb{R}^{n_2 \times n_2}$ is a nonsingular matrix, other matrix blocks have suitable dimensions, $x_{1,s} \in \mathbb{R}^{n_1}$ and $x_{2,s} \in \mathbb{R}^{n_2}$.

The systems (1) and (2) can be rewritten as follows:

$$x_{1,s+1} = A_{1,s}x_{1,s} + B_{1,s}\varpi_s, \quad s \geq 0, \tag{5}$$

$$x_{2,s} = A_{2,s}x_{1,s} + B_{2,s}\varpi_s, \quad s \geq 0, \tag{6}$$

$$z_{i,s} = H_{i,s}x_{1,s} + f_i(x_s, \xi_{i,s}) + \eta_{i,s}, \quad s \geq 0, \tag{7}$$

where

$$A_{1,s} = \Phi_{11,s} - \Phi_{12,s}\Phi_{22,s}^{-1}\Phi_{21,s},\ A_{2,s} = -\Phi_{22,s}^{-1}\Phi_{21,s},\ B_{1,s} = \Upsilon_{1,s} - \Phi_{12,s}\Phi_{22,s}^{-1}\Upsilon_{2,s},$$
$$B_{2,s} = -\Phi_{22,s}^{-1}\Upsilon_{2,s},\ H_{i,s} = C_{i,s}^{(1)} - C_{i,s}^{(2)}\Phi_{22,s}^{-1}\Phi_{21,s},\ F_{i,s} = -C_{i,s}^{(2)}\Phi_{22,s}^{-1}\Upsilon_{2,s},\ \eta_{i,s} = F_{i,s}\varpi_s + \vartheta_{i,s}.$$

The noise sequences ϖ_s and $\eta_{i,s}$ obey

$$\mathbb{E}\left\{\eta_{i,s}\right\} = 0, \quad \mathbb{E}\left\{\begin{bmatrix}\varpi_s \\ \eta_{i,s}\end{bmatrix}\begin{bmatrix}\varpi_h^T & \eta_{j,h}^T\end{bmatrix}\right\} = \begin{bmatrix}Q_{\varpi,s} & R_{j,s} \\ R_{i,s}^T & Q_{\eta,s}^{ij}\end{bmatrix}\delta_{sh},$$

especially when $i = j$, we define $Q_{\eta,s}^{ii} = Q_{\eta,s}^{i}$, where

$$R_{i,s} = \mathbb{E}\left\{\varpi_s\eta_{i,s}^T\right\} = Q_{\varpi,s}F_{i,s}^T,$$
$$Q_{\eta,s}^{i} = \mathbb{E}\left\{\eta_{i,s}\eta_{i,s}^T\right\} = F_{i,s}Q_{\varpi,s}F_{i,s}^T + Q_{\vartheta,s}^{i},$$
$$Q_{\eta,s}^{ij} = \mathbb{E}\left\{\eta_{i,s}\eta_{j,s}^T\right\} = F_{i,s}Q_{\varpi,s}F_{j,s}^T, \quad i \neq j.$$

Lemma 1. *For systems (1) and (2) under Assumptions 1–6, the second moment $\varrho_{s+1} = \mathbb{E}\left\{x_{s+1}x_{s+1}^T\right\}$ can be calculated by:*

$$\varrho_{s+1} = \mathbb{E}\left\{x_{s+1}x_{s+1}^T\right\} = V\begin{bmatrix}\mathbb{E}\left\{x_{1,s+1}x_{1,s+1}^T\right\} & \mathbb{E}\left\{x_{1,s+1}x_{2,s+1}^T\right\} \\ \mathbb{E}\left\{x_{2,s+1}x_{1,s+1}^T\right\} & \mathbb{E}\left\{x_{2,s+1}x_{2,s+1}^T\right\}\end{bmatrix}V^T, \tag{8}$$

where

$$\varrho_{1,s+1} = \mathbb{E}\left\{x_{1,s+1}x_{1,s+1}^T\right\} = A_{1,s}\varrho_{1,s}A_{1,s}^T + B_{1,s}Q_{\varpi,s}B_{1,s}^T, \tag{9}$$

$$\varrho_{12,s+1} = \mathbb{E}\left\{x_{1,s+1}x_{2,s+1}^T\right\} = \varrho_{1,s+1}A_{2,s+1}^T, \tag{10}$$

$$\varrho_{2,s+1} = \mathbb{E}\left\{x_{2,s+1}x_{2,s+1}^T\right\} = A_{2,s+1}\varrho_{1,s+1}A_{2,s+1}^T + B_{2,s+1}Q_{\varpi,s+1}B_{2,s+1}^T. \tag{11}$$

The initial value is $\varrho_0 = \pi_0\pi_0^T + P_0$.

Proof. See Appendix A.1. ■

3. Main results

In this section, our aim is to design the DFF for singular systems with RTDs and PD compensations. Firstly, we introduce the new augmented systems with correlated stochastic parameter matrices and correlated noises. Then, the LFs with corresponding estimation error covariance matrices can be obtained for the augmented systems by utilizing the innovation analysis method. Finally, the DFF weighted by matrices is designed for the original singular systems based on the state transformation.

3.1. Model transformation

Letting $\phi_{i,s} = (1 - \alpha_{i,s})\beta_{i,s+1}$, the measurement equation (3) can be rewritten as:

$$\mathcal{Y}_{i,s} = \alpha_{i,s}z_{i,s} + (1 - \alpha_{i,s})\left\{\phi_{i,s-1}z_{i,s-1} + (1 - \phi_{i,s-1})\hat{z}_{i,s|s-1}\right\}, \quad s \geq 1; \quad \mathcal{Y}_{i,0} = z_{i,0}.$$

Then, defining $\mathcal{Z}_{i,s-1} = \phi_{i,s-1}z_{i,s-1} + (1 - \phi_{i,s-1})\hat{z}_{i,s|s-1}$, we get

$$\mathcal{Y}_{i,s} = \alpha_{i,s}z_{i,s} + (1 - \alpha_{i,s})\mathcal{Z}_{i,s-1}$$
$$= \alpha_{i,s}H_{i,s}x_{1,s} + \alpha_{i,s}f_i(x_s, \xi_{i,s}) + \alpha_{i,s}\eta_{i,s} + (1 - \alpha_{i,s})\mathcal{Z}_{i,s-1}, \quad s \geq 1;$$
$$\mathcal{Y}_{i,0} = z_{i,0}.$$

By using the projection theory, the one-step predictor $\hat{x}_{i,s+1|s}^{(1)}$ for state $x_{1,s}$ and $\hat{z}_{i,s+1|s}$ are given as follows:

$$\hat{x}_{i,s+1|s}^{(1)} = A_{1,s}\hat{x}_{i,s|s-1}^{(1)} + K_{i,s+1|s}\varepsilon_{i,s}, \quad s \geq 0; \quad \hat{x}_{i,0|-1}^{(1)} = [I_{n_1}\ 0]V^{-1}\pi_0, \tag{12}$$

$$\hat{z}_{i,s+1|s} = H_{i,s+1}\hat{x}_{i,s+1|s}^{(1)}, \quad s \geq 0, \tag{13}$$

the gain $K_{i,s+1|s}$ is defined by:

$$K_{i,s+1|s} = \mathbb{E}\left\{x_{1,s+1}\varepsilon_{i,s}^T\right\}\Xi_{i,s}^{-1},$$

where $\varepsilon_{i,s}$ is an innovation with covariance matrix $\Xi_{i,s} = \mathbb{E}\left\{\varepsilon_{i,s}\varepsilon_{i,s}^T\right\}$. According to (7), (12) and (13), $\mathcal{Z}_{i,s}$ can be rewritten as:

$$\mathcal{Z}_{i,s} = \phi_{i,s}H_{i,s}x_{1,s} + \phi_{i,s}f_i(x_s, \xi_{i,s}) + \phi_{i,s}\eta_{i,s} + (1-\phi_{i,s})H_{i,s+1}A_{1,s}\hat{x}_{i,s|s-1}^{(1)} + (1-\phi_{i,s})H_{i,s+1}K_{i,s+1|s}\varepsilon_{i,s}.$$

For the convenience of the subsequent expression, we introduce the following new augmented systems:

$$\mathcal{X}_{i,s+1}^{(1)} = A_{i,s}^{(1)}\mathcal{X}_{i,s}^{(1)} + B_{i,s}^{(1)}W_{i,s} + \phi_{i,s}E_if_i(x_s, \xi_{i,s}) + (1-\phi_{i,s})\Phi_{i,s}\varepsilon_{i,s} + (1-\phi_{i,s})D_{i,s}\hat{\mathcal{X}}_{i,s|s-1}^{(1)}, \quad s \geq 0, \tag{14}$$

$$\mathcal{X}_{i,s}^{(2)} = A_{i,s}^{(2)}\mathcal{X}_{i,s}^{(1)} + B_{i,s}^{(2)}W_{i,s} + \phi_{i,s}E_if_i(x_s, \xi_{i,s}) + (1-\phi_{i,s})\Phi_{i,s}\varepsilon_{i,s} + (1-\phi_{i,s})D_{i,s}\hat{\mathcal{X}}_{i,s|s-1}^{(1)}, \quad s \geq 0, \tag{15}$$

$$\mathcal{Y}_{i,s} = \breve{H}_{i,s}\mathcal{X}_{i,s}^{(1)} + \alpha_{i,s}f_i(x_s, \xi_{i,s}) + \alpha_{i,s}\eta_{i,s}, \quad s \geq 0, \tag{16}$$

where

$$\mathcal{X}_{i,s}^{(1)} = \begin{bmatrix} x_{1,s} \\ \mathcal{Z}_{i,s-1} \end{bmatrix}, \quad \mathcal{X}_{i,s}^{(2)} = \begin{bmatrix} x_{2,s} \\ \mathcal{Z}_{i,s} \end{bmatrix}, \quad W_{i,s} = \begin{bmatrix} \varpi_s \\ \eta_{i,s} \end{bmatrix}, \quad A_{i,s}^{(1)} = \begin{bmatrix} A_{1,s} & 0 \\ \phi_{i,s}H_{i,s} & 0 \end{bmatrix},$$

$$A_{i,s}^{(2)} = \begin{bmatrix} A_{2,s} & 0 \\ \phi_{i,s}H_{i,s} & 0 \end{bmatrix}, \quad B_{i,s}^{(1)} = \begin{bmatrix} B_{1,s} & 0 \\ 0 & \phi_{i,s}I_{m_i} \end{bmatrix}, \quad B_{i,s}^{(2)} = \begin{bmatrix} B_{2,s} & 0 \\ 0 & \phi_{i,s}I_{m_i} \end{bmatrix},$$

$$E_i = \begin{bmatrix} 0 \\ I_{m_i} \end{bmatrix}, \quad \Phi_{i,s} = \begin{bmatrix} 0 \\ H_{i,s+1}K_{i,s+1|s} \end{bmatrix}, \quad D_{i,s} = \begin{bmatrix} 0 & 0 \\ H_{i,s+1}A_{1,s} & 0 \end{bmatrix},$$

$$\breve{H}_{i,s} = \begin{bmatrix} \alpha_{i,s}H_{i,s} & (1-\alpha_{i,s})I_{m_i} \end{bmatrix}.$$

The noise sequences $W_{i,s}$ and $\eta_{j,s}$ obey

$$Q_{W,s}^{ij} = \mathbb{E}\left\{W_{i,s}W_{j,h}^T\right\} = \begin{bmatrix} Q_{\varpi,s} & R_{j,s} \\ R_{i,s}^T & Q_{\eta,s}^{ij} \end{bmatrix}\delta_{sh}, \quad S_{ij,s} = \mathbb{E}\left\{W_{i,s}\eta_{j,h}^T\right\} = \begin{bmatrix} R_{j,s} \\ Q_{\eta,s}^{ij} \end{bmatrix}\delta_{sh},$$

especially when $i = j$, we have $Q_{W,s}^{i} = Q_{W,s}^{ii}$ and $S_{i,s} = S_{ii,s}$. Then, we define

$$\bar{A}_{1,s} = \begin{bmatrix} A_{1,s} & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_{2,s} = \begin{bmatrix} A_{2,s} & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{B}_{1,s} = \begin{bmatrix} B_{1,s} & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{B}_{2,s} = \begin{bmatrix} B_{2,s} & 0 \\ 0 & 0 \end{bmatrix},$$

$$\bar{H}_{i,s} = \begin{bmatrix} 0 & 0 \\ H_{i,s} & 0 \end{bmatrix}, \quad \bar{E}_i = \begin{bmatrix} 0 & 0 \\ 0 & I_{m_i} \end{bmatrix}, \quad \vec{H}_{i,s} = \begin{bmatrix} H_{i,s} & 0 \end{bmatrix}, \quad \vec{E}_i = \begin{bmatrix} 0 & I_{m_i} \end{bmatrix}.$$

For the sake of convenience, the stochastic parameter matrices of systems (14), (15) and (16) can be rewritten as:

$$A_{i,s}^{(1)} = \bar{A}_{1,s} + \phi_{i,s}\bar{H}_{i,s}, \quad A_{i,s}^{(2)} = \bar{A}_{2,s} + \phi_{i,s}\bar{H}_{i,s},$$

$$B_{i,s}^{(1)} = \bar{B}_{1,s} + \phi_{i,s}\bar{E}_i, \quad B_{i,s}^{(2)} = \bar{B}_{2,s} + \phi_{i,s}\bar{E}_i,$$

$$\breve{H}_{i,s} = \alpha_{i,s}\vec{H}_{i,s} + (1-\alpha_{i,s})\vec{E}_i.$$

We readily have expectations

$$\bar{A}_{i,s}^{(1)} = \mathbb{E}\left\{A_{i,s}^{(1)}\right\} = \bar{A}_{1,s} + \bar{\phi}_i\bar{H}_{i,s}, \quad \bar{A}_{i,s}^{(2)} = \mathbb{E}\left\{A_{i,s}^{(2)}\right\} = \bar{A}_{2,s} + \bar{\phi}_i\bar{H}_{i,s},$$

$$\bar{B}_{i,s}^{(1)} = \mathbb{E}\left\{B_{i,s}^{(1)}\right\} = \bar{B}_{1,s} + \bar{\phi}_i\bar{E}_i, \quad \bar{B}_{i,s}^{(2)} = \mathbb{E}\left\{B_{i,s}^{(2)}\right\} = \bar{B}_{2,s} + \bar{\phi}_i\bar{E}_i,$$

$$\bar{\breve{H}}_{i,s} = \mathbb{E}\left\{\breve{H}_{i,s}\right\} = \bar{\alpha}_i\vec{H}_{i,s} + (1-\bar{\alpha}_i)\vec{E}_i, \quad \bar{\phi}_i = \mathbb{E}\left\{\phi_{i,s}\right\} = (1-\bar{\alpha}_i)\bar{\beta}_i.$$

Further, we have

$$A_{i,s}^{(1)} - \bar{A}_{i,s}^{(1)} = A_{i,s}^{(2)} - \bar{A}_{i,s}^{(2)} = (\phi_{i,s} - \bar{\phi}_i)\bar{H}_{i,s},$$

$$B_{i,s}^{(1)} - \bar{B}_{i,s}^{(1)} = B_{i,s}^{(2)} - \bar{B}_{i,s}^{(2)} = (\phi_{i,s} - \bar{\phi}_i)\bar{E}_i,$$

$$\breve{H}_{i,s} - \bar{\breve{H}}_{i,s} = (\alpha_{i,s} - \bar{\alpha}_i)(\vec{H}_{i,s} - \vec{E}_i).$$

Remark 3. Based on the definition of $\phi_{i,s}$, and the distributions of $\alpha_{i,s}$ and $\beta_{i,s}$, we have the following statistical properties:

$$\mathbb{E}\left\{\alpha_{i,s}\right\} = \bar{\alpha}_i, \quad \mathbb{E}\left\{\alpha_{i,s}\phi_{i,s}^2\right\} = 0, \quad \mathbb{E}\left\{\alpha_{i,s}\phi_{i,s}\right\} = 0, \quad \mathbb{E}\left\{(1-\phi_{i,s})^2\right\} = 1-\bar{\phi}_i,$$

$$\mathbb{E}\left\{\alpha_{i,s}^2\right\} = \bar{\alpha}_i, \quad \mathbb{E}\left\{\alpha_{i,s}(1-\phi_{i,s})\right\} = \bar{\alpha}_i, \quad \mathbb{E}\left\{(\alpha_{i,s}-\bar{\alpha}_i)^2\right\} = \bar{\alpha}_i - \bar{\alpha}_i^2, \quad \mathbb{E}\left\{\alpha_{i,s}^2(1-\phi_{i,s})^2\right\} = \bar{\alpha}_i,$$

$$\mathbb{E}\left\{\phi_{i,s}^2\right\} = \bar{\phi}_i, \quad \mathbb{E}\left\{(\alpha_{i,s}-\bar{\alpha}_i)\phi_{i,s}\right\} = -\bar{\alpha}_i\bar{\phi}_i, \quad \mathbb{E}\left\{\alpha_{i,s}\phi_{i,s}(1-\phi_{i,s})\right\} = 0, \quad \mathbb{E}\left\{\alpha_{i,s}(1-\phi_{i,s})^2\right\} = \bar{\alpha}_i,$$

$$\mathbb{E}\left\{(\alpha_{i,s}-\bar{\alpha}_i)(1-\phi_{i,s})\right\} = \bar{\alpha}_i\bar{\phi}_i, \quad \mathbb{E}\left\{(\alpha_{i,s}-\bar{\alpha}_i)\phi_{i,s}(1-\phi_{i,s})\right\} = 0, \quad \mathbb{E}\left\{(\phi_{i,s}-\bar{\phi}_i)^2\right\} = \bar{\phi}_i - \bar{\phi}_i^2,$$

$$\mathbb{E}\left\{(\alpha_{i,s}-\bar{\alpha}_i)(1-\phi_{i,s})^2\right\} = \bar{\alpha}_i\bar{\phi}_i, \quad \mathbb{E}\left\{(\alpha_{i,s}-\bar{\alpha}_i)(\phi_{i,s}-\bar{\phi}_i)\right\} = -\bar{\alpha}_i\bar{\phi}_i, \quad \mathbb{E}\left\{(1-\phi_{i,s})\alpha_{i,s}^2\right\} = \bar{\alpha}_i.$$

It is worth noting that the statistical properties between random variables $\alpha_{i,s}$ and $\phi_{i,s}$ would be used in the subsequent derivation.

Remark 4. It is noteworthy to see that the measurement model (3) includes random variables $\alpha_{i,s}$ and $\beta_{i,s}$, delay measurement $z_{i,s}$ and one-step predictor $\hat{z}_{i,s|s-1}$, which will complicate the derivation of corresponding terms to certain extent. To be more specific, based on the received measurements $\{\mathcal{Y}_{i,s},\ldots,\mathcal{Y}_{i,0}\}$, we can see that it is difficult to directly design the local optimal filters for the reduced-order states $x_{1,s}$ and $x_{2,s}$ through employing the projection theory. In order to attenuate the complexity of the calculation and reduce the computation burden, the new variables $\phi_{i,s}$ and $\mathcal{Z}_{i,s}$ are introduced to convert the original systems (1)–(3) with RTDs and PD compensations into the equivalent augmented systems (14)–(16) with stochastic parameter matrices. Afterwards, on the basis of the new augmented systems (14)–(16), the local optimal filters can be easily obtained according to the projection property in subsequent sections.

3.2. Preliminary lemmas

Lemma 2. *For the augmented subsystems (14), (15) and (16), the innovation sequence $\varepsilon_{i,s}$ is calculated by:*

$$\varepsilon_{i,s}=(\alpha_{i,s}-\bar{\alpha}_i)(\vec{H}_{i,s}-\vec{E}_i)\mathcal{X}_{i,s}^{(1)}+\alpha_{i,s}f_i(x_s,\xi_{i,s})+\alpha_{i,s}\eta_{i,s}+\bar{\vec{H}}_{i,s}\tilde{\mathcal{X}}_{i,s|s-1}^{(1)},\quad s\geq 0. \tag{17}$$

The innovation covariance $\Xi_{i,s}=\mathbb{E}\{\varepsilon_{i,s}\varepsilon_{i,s}^T\}$ is calculated by:

$$\Xi_{i,s}=(\bar{\alpha}_i-\bar{\alpha}_i^2)(\vec{H}_{i,s}-\vec{E}_i)q_{i,s}^{(1)}(\vec{H}_{i,s}-\vec{E}_i)^T+\bar{\alpha}_i\sum_{l=1}^{m}\Pi_i^{(l)}\mathrm{tr}(\varrho_s\Gamma_i^{(l)})+\bar{\alpha}_iQ_{\eta,s}^i+\bar{\vec{H}}_{i,s}\mathcal{P}_{i,s|s-1}^{(1)}\bar{\vec{H}}_{i,s}^T,\quad s\geq 0. \tag{18}$$

The innovation cross-covariance $\Xi_{ij,s}=\mathbb{E}\{\varepsilon_{i,s}\varepsilon_{j,s}^T\}(i\neq j)$ between $\varepsilon_{i,s}$ and $\varepsilon_{j,s}$ is computed by:

$$\Xi_{ij,s}=\bar{\alpha}_i\bar{\alpha}_jQ_{\eta,s}^{ij}+\bar{\vec{H}}_{i,s}\mathcal{P}_{ij,s|s-1}^{(1)}\bar{\vec{H}}_{j,s}^T,\quad s\geq 0. \tag{19}$$

In the Eqs. (18) and (19), the second-order moment $q_{i,s}^{(1)}=\mathbb{E}\{\mathcal{X}_{i,s}^{(1)}\mathcal{X}_{i,s}^{(1)T}\}$, the prediction error covariance $\mathcal{P}_{i,s|s-1}^{(1)}=\mathbb{E}\{\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}\tilde{\mathcal{X}}_{i,s|s-1}^{(1)T}\}$ and the prediction error cross-covariance $\mathcal{P}_{ij,s|s-1}^{(1)}=\mathbb{E}\{\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}\tilde{\mathcal{X}}_{j,s|s-1}^{(1)T}\}$ can be computed by the subsequent Lemma 3 and Theorem 2, respectively.

Proof. See Appendix A.2. ■

Lemma 3. *For the augmented subsystems (14), (15) and (16), the second-order moment $q_{i,s}^{(1)}=\mathbb{E}\{\mathcal{X}_{i,s}^{(1)}\mathcal{X}_{i,s}^{(1)T}\}$ for the state $\mathcal{X}_{i,s}^{(1)}$ satisfies*

$$\begin{aligned}q_{i,s+1}^{(1)}&=\bar{A}_{1,s}q_{i,s}^{(1)}\bar{A}_{1,s}^T+\bar{\phi}_i\bar{H}_{i,s}q_{i,s}^{(1)}\bar{H}_{i,s}^T+\bar{\phi}_{i,s}\bar{A}_{1,s}q_{i,s}^{(1)}\bar{H}_{i,s}^T+\bar{\phi}_i\bar{H}_{i,s}q_{i,s}^{(1)}\bar{A}_{1,s}^T+\bar{B}_{1,s}Q_{W,s}^i\bar{B}_{1,s}^T\\&\quad+\bar{\phi}_{i,s}\bar{E}_iQ_{W,s}^i\bar{E}_i^T+\bar{\phi}_i\bar{B}_{1,s}Q_{W,s}^i\bar{E}_i^T+\bar{\phi}_i\bar{E}_iQ_{W,s}^i\bar{B}_{1,s}^T+\bar{\phi}_iE_i\sum_{l=1}^{m}\Pi_i^{(l)}\mathrm{tr}(\varrho_s\Gamma_i^{(l)})E_i^T\\&\quad+\mathfrak{A}_{i1,s}+(1-\bar{\phi}_i)D_{i,s}(q_{i,s}^{(1)}-\mathcal{P}_{i,s|s-1}^{(1)})D_{i,s}^T+\Big\{\mathfrak{B}_{i1,s}+\mathfrak{B}_{i2,s}+\mathfrak{B}_{i3,s}+\mathfrak{B}_{i4,s}\Big\}+\Big\{*\Big\}^T,\end{aligned} \tag{20}$$

where

$$\begin{aligned}\mathfrak{A}_{i1,s}&=(\bar{\alpha}_i-\bar{\alpha}_i^2(1+\bar{\phi}_i))\Phi_{i,s}(\vec{H}_{i,s}-\vec{E}_i)q_{i,s}^{(1)}(\vec{H}_{i,s}-\vec{E}_i)^T\Phi_{i,s}^T+\bar{\alpha}_i\Phi_{i,s}\sum_{l=1}^{m}\Pi_i^{(l)}\mathrm{tr}(\varrho_s\Gamma_i^{(l)})\Phi_{i,s}^T\\&\quad+\bar{\alpha}_i\Phi_{i,s}Q_{\eta,s}^i\Phi_{i,s}^T+(1-\bar{\phi}_i)\Phi_{i,s}\bar{\vec{H}}_{i,s}\mathcal{P}_{i,s|s-1}^{(1)}\bar{\vec{H}}_{i,s}^T\Phi_{i,s}^T+\Big\{\bar{\alpha}_i\bar{\phi}_i\Phi_{i,s}(\vec{H}_{i,s}-\vec{E}_i)\mathcal{P}_{i,s|s-1}^{(1)}\bar{\vec{H}}_{i,s}^T\Phi_{i,s}^T\Big\}\\&\quad+\Big\{*\Big\}^T,\\\mathfrak{B}_{i1,s}&=\bar{\alpha}_i\bar{\phi}_i\bar{A}_{1,s}q_{i,s}^{(1)}(\vec{H}_{i,s}-\vec{E}_i)^T\Phi_{i,s}^T+(1-\bar{\phi}_i)\bar{A}_{1,s}\mathcal{P}_{i,s|s-1}^{(1)}\bar{\vec{H}}_{i,s}^T\Phi_{i,s}^T,\\\mathfrak{B}_{i2,s}&=(1-\bar{\phi}_i)\bar{A}_{1,s}\big(q_{i,s}^{(1)}-\mathcal{P}_{i,s|s-1}^{(1)}\big)D_{i,s}^T,\\\mathfrak{B}_{i3,s}&=\bar{\alpha}_i\bar{B}_{1,s}S_{i,s}\Phi_{i,s}^T,\\\mathfrak{B}_{i4,s}&=\bar{\alpha}_i\bar{\phi}_i\Phi_{i,s}(\vec{H}_{i,s}-\vec{E}_i)\big(q_{i,s}^{(1)}-\mathcal{P}_{i,s|s-1}^{(1)}\big)D_{i,s}^T.\end{aligned}$$

The cross-state second-order moment matrix $q_{ij,s}^{(1)}=\mathbb{E}\{\mathcal{X}_{i,s}^{(1)}\mathcal{X}_{j,s}^{(1)T}\}(i\neq j)$ for states $\mathcal{X}_{i,s}^{(1)}$ and $\mathcal{X}_{j,s}^{(1)}$ satisfies

$$q_{ij,s+1}^{(1)}=\bar{A}_{i,s}^{(1)}q_{ij,s}^{(1)}\bar{A}_{j,s}^{(1)T}+\bar{B}_{i,s}^{(1)}Q_{W,s}^{ij}\bar{B}_{j,s}^{(1)T}+\mathfrak{A}_{ij1,s}+(1-\bar{\phi}_i)(1-\bar{\phi}_j)D_{i,s}\big(q_{ij,s}^{(1)}+\mathcal{P}_{ij,s|s-1}^{(1)}-\bar{\mathcal{P}}_{ij,s|s|s-1}^{(1)}$$

$$-\bar{\mathcal{P}}^{(1)T}_{ji,s|s|s-1}\big)D^T_{j,s}+\mathfrak{B}_{ij1,s}+\mathfrak{B}^T_{ji1,s}+\mathfrak{B}_{ij2,s}+\mathfrak{B}^T_{ji2,s}+\mathfrak{B}_{ij3,s}+\mathfrak{B}^T_{ji3,s}+\mathfrak{B}_{ij4,s}+\mathfrak{B}^T_{ji4,s}, \tag{21}$$

where

$$\begin{aligned}
\mathfrak{A}_{ij1,s} &= \bar{\alpha}_i\bar{\phi}_i\bar{\alpha}_j\bar{\phi}_j\Phi_{i,s}(\vec{H}_{i,s}-\vec{E}_i)q^{(1)}_{ij,s}(\vec{H}_{j,s}-\vec{E}_j)^T\Phi^T_{j,s}+\bar{\alpha}_i\bar{\alpha}_j\Phi_{i,s}Q^{ij}_{\eta,s}\Phi^T_{j,s}+(1-\bar{\phi}_i)(1-\bar{\phi}_j)\Phi_{i,s}\breve{\bar{H}}_{i,s}\\
&\quad\times\mathcal{P}^{(1)}_{ij,s|s-1}\breve{\bar{H}}^T_{j,s}\Phi^T_{j,s}+\bar{\alpha}_i\bar{\phi}_i(1-\bar{\phi}_j)\Phi_{i,s}(\vec{H}_{i,s}-\vec{E}_i)\bar{\mathcal{P}}^{(1)}_{ij,s|s|s-1}\breve{\bar{H}}^T_{j,s}\Phi^T_{j,s}+\bar{\alpha}_j\bar{\phi}_j(1-\bar{\phi}_i)\Phi_{i,s}\breve{\bar{H}}_{i,s}\\
&\quad\times\bar{\mathcal{P}}^{(1)T}_{ji,s|s|s-1}(\vec{H}_{j,s}-\vec{E}_j)^T\Phi^T_{j,s},\\
\mathfrak{B}_{ij1,s} &= \bar{\alpha}_j\bar{\phi}_j\bar{A}^{(1)}_{i,s}q^{(1)}_{ij,s}(\vec{H}_{j,s}-\vec{E}_j)^T\Phi^T_{j,s}+(1-\bar{\phi}_j)\bar{A}^{(1)}_{i,s}\bar{\mathcal{P}}^{(1)}_{ij,s|s|s-1}\breve{\bar{H}}^T_{j,s}\Phi^T_{j,s},\\
\mathfrak{B}_{ij2,s} &= (1-\bar{\phi}_j)\bar{A}^{(1)}_{i,s}\big(q^{(1)}_{ij,s}-\bar{\mathcal{P}}^{(1)}_{ij,s|s|s-1}\big)D^T_{j,s},\\
\mathfrak{B}_{ij3,s} &= \bar{\alpha}_j\bar{B}^{(1)}_{i,s}S_{ij,s}\Phi^T_{j,s},\\
\mathfrak{B}_{ij4,s} &= \bar{\alpha}_i\bar{\phi}_i(1-\bar{\phi}_j)\Phi_{i,s}(\vec{H}_{i,s}-\vec{E}_i)\big(q^{(1)}_{ij,s}-\bar{\mathcal{P}}^{(1)}_{ij,s|s|s-1}\big)D^T_{j,s}+(1-\bar{\phi}_i)(1-\bar{\phi}_j)\Phi_{i,s}\breve{\bar{H}}_{i,s}\big(\bar{\mathcal{P}}^{(1)T}_{ji,s|s|s-1}\\
&\quad-\mathcal{P}^{(1)}_{ij,s|s-1}\big)D^T_{j,s}.
\end{aligned}$$

The initial value is $q^{(1)}_{i,s}=q^{(1)}_{ij,s}=\mathrm{diag}\{[I_{n_1}\ \ 0]V^{-1}(\pi_0\pi_0^T+P_0)(V^T)^{-1}[I_{n_1}\ \ 0]^T,\ \ 0\}$. *In Eq. (21), the covariance* $\bar{\mathcal{P}}^{(1)}_{ij,s|s|s-1}=\mathbb{E}\{\mathcal{X}^{(1)}_{i,s}\tilde{\mathcal{X}}^{(1)T}_{j,s|s-1}\}$ *can be calculated by the following Lemma 4.*

Proof. See Appendix A.3 ■

3.3. DFFs for the augmented systems

3.3.1. Design of LFs

Theorem 1. *For augmented subsystems (14), (15) and (16) under Assumptions 4–6, the LFs and one-step predictor for states* $\mathcal{X}^{(1)}_{i,s}$ *and* $\mathcal{X}^{(2)}_{i,s}$ *are given by:*

$$\hat{\mathcal{X}}^{(1)}_{i,s|s} = \hat{\mathcal{X}}^{(1)}_{i,s|s-1}+G_{i,s|s}\varepsilon_{i,s},\quad s\geq 0, \tag{22}$$

$$\hat{\mathcal{X}}^{(1)}_{i,s+1|s} = \big(\bar{A}^{(1)}_{i,s}+(1-\bar{\phi}_i)D_{i,s}\big)\hat{\mathcal{X}}^{(1)}_{i,s|s-1}+G_{i,s+1|s}\varepsilon_{i,s},\quad s\geq 0;\quad \hat{\mathcal{X}}^{(1)}_{i,0|-1}=\Big[\big([I_{n_1}\ 0]V^{-1}\pi_0\big)^T\quad 0^T\Big]^T, \tag{23}$$

$$\hat{\mathcal{X}}^{(2)}_{i,s|s} = \big(\bar{A}^{(2)}_{i,s}+(1-\bar{\phi}_i)D_{i,s}\big)\hat{\mathcal{X}}^{(1)}_{i,s|s-1}+L_{i,s|s}\varepsilon_{i,s},\quad s\geq 0, \tag{24}$$

where $\varepsilon_{i,s}$ *is the innovation sequence,* $G_{i,s|s}$, $G_{i,s+1|s}$ *and* $L_{i,s|s}$ *are the gain matrices. They are calculated by the subsequent text, respectively.*

The gain matrices $G_{i,s|s}$, $G_{i,s+1|s}$ *and* $L_{i,s|s}$ *are, respectively, calculated by:*

$$G_{i,s|s} = \mathcal{P}^{(1)}_{i,s|s-1}\breve{\bar{H}}^T_{i,s}\Xi^{-1}_{i,s}, \tag{25}$$

$$\begin{aligned}
G_{i,s+1|s} &= \Big\{\bar{A}^{(1)}_{i,s}\mathcal{P}^{(1)}_{i,s|s-1}\breve{\bar{H}}^T_{i,s}-\bar{\alpha}_i\bar{\phi}_i\bar{H}_{i,s}q^{(1)}_{i,s}(\vec{H}_{i,s}-\vec{E}_i)^T+\bar{\alpha}_i\bar{B}_{1,s}S_{i,s}+\mathfrak{C}_{i1,s}+\bar{\alpha}_i\bar{\phi}_iD_{i,s}\big(q^{(1)}_{i,s}\\
&\quad-\mathcal{P}^{(1)}_{i,s|s-1}\big)(\vec{H}_{i,s}-\vec{E}_i)^T\Big\}\Xi^{-1}_{i,s},
\end{aligned}\tag{26}$$

$$\begin{aligned}
L_{i,s|s} &= \Big\{\bar{A}^{(2)}_{i,s}\mathcal{P}^{(1)}_{i,s|s-1}\breve{\bar{H}}^T_{i,s}-\bar{\alpha}_i\bar{\phi}_i\bar{H}_{i,s}q^{(1)}_{i,s}(\vec{H}_{i,s}-\vec{E}_i)^T+\bar{\alpha}_i\bar{B}_{2,s}S_{i,s}+\mathfrak{C}_{i1,s}+\bar{\alpha}_i\bar{\phi}_iD_{i,s}\big(q^{(1)}_{i,s}\\
&\quad-\mathcal{P}^{(1)}_{i,s|s-1}\big)(\vec{H}_{i,s}-\vec{E}_i)^T\Big\}\Xi^{-1}_{i,s},
\end{aligned}\tag{27}$$

where

$$\begin{aligned}
\mathfrak{C}_{i1,s} &= \big(\bar{\alpha}_i-\bar{\alpha}_i^2(1+\bar{\phi}_i)\big)\Phi_{i,s}(\vec{H}_{i,s}-\vec{E}_i)q^{(1)}_{i,s}(\vec{H}_{i,s}-\vec{E}_i)^T+\bar{\alpha}_i\Phi_{i,s}\sum_{l=1}^{m}\Pi^{(l)}_i\mathrm{tr}(\varrho_s\Gamma^{(l)}_i)\\
&\quad+\bar{\alpha}_i\Phi_{i,s}Q^i_{\eta,s}+(1-\bar{\phi}_i)\Phi_{i,s}\breve{\bar{H}}_{i,s}\mathcal{P}^{(1)}_{i,s|s-1}\breve{\bar{H}}^T_{i,s}+\bar{\alpha}_i\bar{\phi}_i\Phi_{i,s}(\vec{H}_{i,s}-\vec{E}_i)\mathcal{P}^{(1)}_{i,s|s-1}\breve{\bar{H}}^T_{i,s}\\
&\quad+\bar{\alpha}_i\bar{\phi}_i\Phi_{i,s}\breve{\bar{H}}_{i,s}\mathcal{P}^{(1)}_{i,s|s-1}(\vec{H}_{i,s}-\vec{E}_i)^T.
\end{aligned}\tag{28}$$

Specially, $\Phi_{i,s}$ *can be rewritten as:*

$$\Phi_{i,s}=D_{i,s}G_{i,s|s}+\begin{bmatrix}0\\ \bar{\alpha}_iH_{i,s+1}B_{1,s}R_{i,s}\Xi^{-1}_{i,s}\end{bmatrix}. \tag{29}$$

Proof. See Appendix A.4 ■

3.3.2. Computation of cross-covariance matrices

Theorem 2. *The filtering error covariance $\mathcal{P}^{(1)}_{i,s|s}$ and the prediction error covariance $\mathcal{P}^{(1)}_{i,s+1|s}$ for state $\mathcal{X}^{(1)}_{i,s}$ are computed as:*

$$\mathcal{P}^{(1)}_{i,s|s} = \mathcal{P}^{(1)}_{i,s|s-1} - G_{i,s|s}\Xi_{i,s}G^T_{i,s|s}, \quad s \geq 0, \tag{30}$$

$$\begin{aligned}
\mathcal{P}^{(1)}_{i,s+1|s} &= \bar{A}_{1,s}\mathcal{P}^{(1)}_{i,s|s-1}\bar{A}^T_{1,s} + \bar{\phi}_i\bar{A}_{1,s}\mathcal{P}^{(1)}_{i,s|s-1}\bar{H}^T_{i,s} + \bar{\phi}_i\bar{H}_{i,s}\mathcal{P}^{(1)}_{i,s|s-1}\bar{A}^T_{1,s} + \bar{\phi}_i\bar{H}_{i,s}\mathcal{P}^{(1)}_{i,s|s-1}\bar{H}^T_{i,s} \\
&\quad + \bar{B}_{1,s}Q^i_{W,s}\bar{B}^T_{1,s} + \bar{\phi}_i\bar{E}_iQ^i_{W,s}\bar{E}_i^T + \bar{\phi}_i\bar{B}_{1,s}Q^i_{W,s}\bar{E}^T_i + \bar{\phi}_i\bar{E}_iQ^i_{W,s}\bar{B}^T_{1,s} \\
&\quad + \bar{\phi}_iE_i\sum_{l=1}^{m}\Pi^{(l)}_i\mathrm{tr}(\varrho_s\Gamma^{(l)}_i)E^T_i + \mathfrak{A}_{i1,s} + (\bar{\phi}_i - \bar{\phi}^2_i)(\bar{H}_{i,s} - D_{i,s})\big(q^{(1)}_{i,s} - \mathcal{P}^{(1)}_{i,s|s-1}\big)(\bar{H}_{i,s} - D_{i,s})^T \\
&\quad + G_{i,s+1|s}\Xi_{i,s}G^T_{i,s+1|s} + \Big\{\mathfrak{D}_{i1,s} - \mathfrak{D}_{i2,s} - \mathfrak{D}_{i3,s} + \mathfrak{D}_{i4,s} - \mathfrak{D}_{i5,s} - \mathfrak{D}_{i6,s} + \mathfrak{B}_{i3,s}\Big\} + \Big\{*\Big\}^T, \quad s \geq 0; \\
\mathcal{P}^{(1)}_{i,0|-1} &= \mathrm{diag}\big\{[I_{n_1}\ 0]V^{-1}P_0(V^T)^{-1}[I_{n_1}\ 0]^T,\ 0\big\},
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
\mathfrak{D}_{i1,s} &= \bar{\alpha}_i\bar{\phi}_i\bar{A}_{1,s}\mathcal{P}^{(1)}_{i,s|s-1}(\vec{H}_{i,s} - \vec{E}_i)^T\Phi^T_{i,s} + (1 - \bar{\phi}_i)\bar{A}_{1,s}\mathcal{P}^{(1)}_{i,s|s-1}\breve{\bar{H}}^T_{i,s}\Phi^T_{i,s}, \\
\mathfrak{D}_{i2,s} &= \bar{A}^{(1)}_{i,s}\mathcal{P}^{(1)}_{i,s|s-1}\breve{\bar{H}}^T_{i,s}G^T_{i,s+1|s} - \bar{\alpha}_i\bar{\phi}_i\bar{H}_{i,s}\mathcal{P}^{(1)}_{i,s|s-1}(\vec{H}_{i,s} - \vec{E}_i)^TG^T_{i,s+1|s}, \\
\mathfrak{D}_{i3,s} &= \bar{\alpha}_i\bar{B}_{1,s}S_{i,s}G^T_{i,s+1|s}, \\
\mathfrak{D}_{i4,s} &= \bar{\alpha}_i\bar{\phi}^2_i\Phi_{i,s}(\vec{H}_{i,s} - \vec{E}_i)\big(\mathcal{P}^{(1)}_{i,s|s-1} - q^{(1)}_{i,s}\big)(\bar{H}_{i,s} - D_{i,s})^T, \\
\mathfrak{D}_{i5,s} &= \mathfrak{C}_{i1,s}G^T_{i,s+1|s}, \\
\mathfrak{D}_{i6,s} &= \bar{\alpha}_i\bar{\phi}_i(\bar{H}_{i,s} - D_{i,s})\big(\mathcal{P}^{(1)}_{i,s|s-1} - q^{(1)}_{i,s}\big)(\vec{H}_{i,s} - \vec{E}_i)^TG^T_{i,s+1|s}.
\end{aligned}$$

The filtering error cross-covariance $\mathcal{P}^{(1)}_{ij,s|s}$ and the prediction error cross-covariance $\mathcal{P}^{(1)}_{ij,s+1|s}$ among the ith and the jth sensor $i \neq j$ for states $\mathcal{X}^{(1)}_{i,s}$ and $\mathcal{X}^{(1)}_{j,s}$ are computed as:

$$\mathcal{P}^{(1)}_{ij,s|s} = \mathcal{P}^{(1)}_{ij,s|s-1} + G_{i,s|s}\Xi_{ij,s}G^T_{j,s|s} - G_{i,s|s}\breve{\bar{H}}_{i,s}\mathcal{P}^{(1)}_{ij,s|s-1} - \mathcal{P}^{(1)}_{ij,s|s-1}\breve{\bar{H}}^T_{j,s}G^T_{j,s|s}, \quad s \geq 0, \tag{32}$$

$$\begin{aligned}
\mathcal{P}^{(1)}_{ij,s+1|s} &= \bar{A}^{(1)}_{i,s}\mathcal{P}^{(1)}_{ij,s|s-1}\bar{A}^{(1)T}_{j,s} + \bar{B}^{(1)}_{i,s}Q^{ij}_{W,s}\bar{B}^{(1)T}_{j,s} + \mathfrak{A}_{ij1,s} + G_{i,s+1|s}\Xi_{ij,s}G^T_{j,s+1|s} + \mathfrak{D}_{ij1,s} + \mathfrak{D}^T_{ji1,s} - \mathfrak{D}_{ij2,s} \\
&\quad - \mathfrak{D}^T_{ji2,s} - \mathfrak{D}_{ij3,s} - \mathfrak{D}^T_{ji3,s} - \mathfrak{D}_{ij4,s} - \mathfrak{D}^T_{ji4,s} + \mathfrak{B}_{ij3,s} + \mathfrak{B}^T_{ji3,s}, \quad s \geq 0; \\
\mathcal{P}^{(1)}_{ij,0|-1} &= \mathrm{diag}\big\{[I_{n_1}\ 0]V^{-1}P_0(V^T)^{-1}[I_{n_1}\ 0]^T,\ 0\big\},
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
\mathfrak{D}_{ij1,s} &= \bar{\alpha}_j\bar{\phi}_j\bar{A}^{(1)}_{i,s}\bar{\mathcal{P}}^{(1)T}_{ji,s|s|s-1}(\vec{H}_{j,s} - \vec{E}_j)^T\Phi^T_{j,s} + (1 - \bar{\phi}_j)\bar{A}^{(1)}_{i,s}\mathcal{P}^{(1)}_{ij,s|s-1}\breve{\bar{H}}^T_{j,s}\Phi^T_{j,s}, \\
\mathfrak{D}_{ij2,s} &= \bar{A}^{(1)}_{i,s}\mathcal{P}^{(1)}_{ij,s|s-1}\breve{\bar{H}}^T_{j,s}G^T_{j,s+1|s}, \\
\mathfrak{D}_{ij3,s} &= \bar{\alpha}_j\bar{B}^{(1)}_{i,s}S_{ij,s}G^T_{j,s+1|s}, \\
\mathfrak{D}_{ij4,s} &= \bar{\alpha}_i\bar{\alpha}_j\Phi_{i,s}Q^{ij}_{\eta,s}G^T_{j,s+1|s} + (1 - \bar{\phi}_i)\Phi_{i,s}\breve{\bar{H}}_{i,s}\mathcal{P}^{(1)}_{ij,s|s-1}\breve{\bar{H}}^T_{j,s}G^T_{j,s+1|s} \\
&\quad + \bar{\alpha}_i\bar{\phi}_i\Phi_{i,s}(\vec{H}_{i,s} - \vec{E}_i)\bar{\mathcal{P}}^{(1)}_{ij,s|s|s-1}\breve{\bar{H}}^T_{j,s}G^T_{j,s+1|s}.
\end{aligned}$$

The filtering error covariance $\mathcal{P}^{(2)}_{i,s|s}$ and the cross-covariance $\mathcal{P}^{(2)}_{ij,s|s}$ among the ith and the jth sensor $(i \neq j)$ for state $\mathcal{X}^{(2)}_{i,s}$ are computed as:

$$\begin{aligned}
\mathcal{P}^{(2)}_{i,s|s} &= \bar{A}_{2,s}\mathcal{P}^{(1)}_{i,s|s-1}\bar{A}^T_{2,s} + \bar{\phi}_i\bar{A}_{2,s}\mathcal{P}^{(1)}_{i,s|s-1}\bar{H}^T_{i,s} + \bar{\phi}_i\bar{H}_{i,s}\mathcal{P}^{(1)}_{i,s|s-1}\bar{A}^T_{2,s} + \bar{\phi}_i\bar{H}_{i,s}\mathcal{P}^{(1)}_{i,s|s-1}\bar{H}^T_{i,s} \\
&\quad + \bar{B}_{2,s}Q^i_{W,s}\bar{B}^T_{2,s} + \bar{\phi}_{i,s}\bar{E}_iQ^i_{W,s}\bar{E}_i^T + \bar{\phi}_i\bar{B}_{2,s}Q^i_{W,s}\bar{E}^T_i + \bar{\phi}_i\bar{E}_iQ^i_{W,s}\bar{B}^T_{2,s} \\
&\quad + \bar{\phi}_iE_i\sum_{l=1}^{m}\Pi^{(l)}_i\mathrm{tr}(\varrho_s\Gamma^{(l)}_i)E^T_i + \mathfrak{A}_{i1,s} + (\bar{\phi}_i - \bar{\phi}^2_i)(\bar{H}_{i,s} - D_{i,s})\big(q^{(1)}_{i,s} - \mathcal{P}^{(1)}_{i,s|s-1}\big)(\bar{H}_{i,s} - D_{i,s})^T \\
&\quad + L_{i,s|s}\Xi_{i,s}L^T_{i,s|s} + \Big\{\mathfrak{E}_{i1,s} - \mathfrak{E}_{i2,s} + \mathfrak{E}_{i3,s} - \mathfrak{E}_{i4,s} - \mathfrak{E}_{i5,s} - \mathfrak{E}_{i6,s} + \mathfrak{D}_{i4,s}\Big\} + \Big\{*\Big\}^T,
\end{aligned} \tag{34}$$

$$\mathcal{P}^{(2)}_{ij,s|s} = \bar{A}^{(1)}_{i,s}\mathcal{P}^{(1)}_{ij,s|s-1}\bar{A}^{(1)T}_{j,s} + \bar{B}^{(1)}_{i,s}Q^{ij}_{W,s}\bar{B}^{(1)T}_{j,s} + \mathfrak{A}_{ij1,s} + L_{i,s|s}\Xi_{ij,s}L^T_{j,s|s} + \mathfrak{E}_{ij1,s} + \mathfrak{E}^T_{ji1,s}$$

$$-\mathfrak{C}_{ij2,s}-\mathfrak{C}_{ji2,s}^{T}+\mathfrak{C}_{ij3,s}+\mathfrak{C}_{ji3,s}^{T}-\mathfrak{C}_{ij4,s}-\mathfrak{C}_{ji4,s}^{T}-\mathfrak{C}_{ij5,s}-\mathfrak{C}_{ji5,s}^{T}, \tag{35}$$

where

$$\begin{aligned}
\mathfrak{C}_{i1,s} &= \bar{\alpha}_i\bar{\phi}_i\bar{A}_{2,s}\mathcal{P}^{(1)}_{i,s|s-1}(\vec{H}_{i,s}-\vec{E}_i)^T\Phi^T_{i,s}+(1-\bar{\phi}_i)\bar{A}_{2,s}\mathcal{P}^{(1)}_{i,s|s-1}\breve{\bar{H}}^T_{i,s}\Phi^T_{i,s},\\
\mathfrak{C}_{i2,s} &= \bar{A}^{(2)}_{i,s}\mathcal{P}^{(1)}_{i,s|s-1}\breve{\bar{H}}^T_{i,s}L^T_{i,s|s}-\bar{\alpha}_i\bar{\phi}_i\bar{H}_{i,s}\mathcal{P}^{(1)}_{i,s|s-1}(\vec{H}_{i,s}-\vec{E}_i)^T L^T_{i,s|s},\\
\mathfrak{C}_{i3,s} &= \bar{\alpha}_i\bar{B}_{2,s}S_{i,s}\Phi^T_{i,s},\\
\mathfrak{C}_{i4,s} &= \bar{\alpha}_i\bar{B}_{2,s}S_{i,s}L^T_{i,s|s},\\
\mathfrak{C}_{i5,s} &= \mathfrak{C}_{i1,s}L^T_{i,s|s},\\
\mathfrak{C}_{i6,s} &= \bar{\alpha}_i\bar{\phi}_i(\bar{H}_{i,s}-D_{i,s})\big(\mathcal{P}^{(1)}_{i,s|s-1}-q^{(1)}_{i,s}\big)(\vec{H}_{i,s}-\vec{E}_i)^T L^T_{i,s|s},\\
\mathfrak{C}_{ij1,s} &= \bar{\alpha}_j\bar{\phi}_j\bar{A}^{(2)}_{i,s}\bar{\mathcal{P}}^{(1)T}_{ji,s|s|s-1}(\vec{H}_{j,s}-\vec{E}_j)^T\Phi^T_{j,s}+(1-\bar{\phi}_j)\bar{A}^{(2)}_{i,s}\mathcal{P}^{(1)}_{ij,s|s-1}\breve{\bar{H}}^T_{j,s}\Phi^T_{j,s},\\
\mathfrak{C}_{ij2,s} &= \bar{A}^{(2)}_{i,s}\mathcal{P}^{(1)}_{ij,s|s-1}\breve{\bar{H}}^T_{j,s}L^T_{j,s|s},\\
\mathfrak{C}_{ij3,s} &= \bar{\alpha}_j\bar{B}^{(2)}_{i,s}S_{ij,s}\Phi^T_{j,s},\\
\mathfrak{C}_{ij4,s} &= \bar{\alpha}_j\bar{B}^{(2)}_{i,s}S_{ij,s}L^T_{j,s|s},\\
\mathfrak{C}_{ij5,s} &= \bar{\alpha}_i\bar{\alpha}_j\Phi_{i,s}Q^{ij}_{\eta,s}L^T_{j,s|s}+(1-\bar{\phi}_i)\Phi_{i,s}\breve{\bar{H}}_{i,s}\mathcal{P}^{(1)}_{ij,s|s-1}\breve{\bar{H}}^T_{j,s}L^T_{j,s|s}\\
&\quad+\bar{\alpha}_i\bar{\phi}_i\Phi_{i,s}(\vec{H}_{i,s}-\vec{E}_i)\bar{\mathcal{P}}^{(1)}_{ij,s|s|s-1}\breve{\bar{H}}^T_{j,s}L^T_{j,s|s}.
\end{aligned}$$

The correlation matrices $\mathcal{P}^{(1,2)}_{i,s|s}$ *and* $\mathcal{P}^{(1,2)}_{ij,s|s}$ $(i\neq j)$ *between* $\tilde{\mathcal{X}}^{(1)}_{i,s|s}$ *and* $\tilde{\mathcal{X}}^{(2)}_{j,s|s}$ *are computed as:*

$$\begin{aligned}
\mathcal{P}^{(1,2)}_{i,s|s} &= \mathcal{P}^{(1)}_{i,s|s-1}\bar{A}^{(2)}_{i,s}+\bar{\alpha}_i\bar{\phi}_i\mathcal{P}^{(1)}_{i,s|s-1}(\vec{H}_{i,s}-\vec{E}_i)^T\Phi^T_{i,s}+(1-\bar{\phi}_i)\mathcal{P}^{(1)}_{i,s|s-1}\breve{\bar{H}}^T_{i,s}\Phi^T_{i,s}-\mathcal{P}^{(1)}_{i,s|s-1}\breve{\bar{H}}^T_{i,s}L^T_{i,s|s}\\
&\quad+\bar{\alpha}_i\bar{\phi}_iG_{i,s|s}(\vec{H}_{i,s}-\vec{E}_i)\mathcal{P}^{(1)}_{i,s|s-1}\breve{\bar{H}}^T_{i,s}-G_{i,s|s}\breve{\bar{H}}_{i,s}\mathcal{P}^{(1)}_{i,s|s-1}\bar{A}^{(2)T}_{i,s}-\bar{\alpha}_iG_{i,s|s}S^T_{i,s}\bar{B}^T_{2,s}-G_{i,s|s}\mathfrak{C}^T_{i1,s}\\
&\quad-\bar{\alpha}_i\bar{\phi}_iG_{i,s|s}(\vec{H}_{i,s}-\vec{E}_i)\big(\mathcal{P}^{(1)}_{i,s|s-1}-q^{(1)}_{i,s}\big)(\bar{H}_{i,s}-D_{i,s})^T+G_{i,s|s}\Xi_{i,s}L^T_{i,s|s},
\end{aligned} \tag{36}$$

$$\begin{aligned}
\mathcal{P}^{(1,2)}_{ij,s|s} &= \mathcal{P}^{(1)}_{ij,s|s-1}\bar{A}^{(2)}_{j,s}+\bar{\alpha}_j\bar{\phi}_j\bar{\mathcal{P}}^{(1)T}_{ji,s|s|s-1}(\vec{H}_{j,s}-\vec{E}_j)^T\Phi^T_{j,s}+(1-\bar{\phi}_j)\mathcal{P}^{(1)}_{ij,s|s-1}\breve{\bar{H}}^T_{j,s}\Phi^T_{j,s}\\
&\quad-G_{i,s|s}\breve{\bar{H}}_{i,s}\mathcal{P}^{(1)}_{ij,s|s-1}\bar{A}^{(2)T}_{j,s}-\bar{\alpha}_iG_{i,s|s}S^T_{ji,s}\bar{B}^{(2)T}_{j,s}-\bar{\alpha}_i\bar{\alpha}_jG_{i,s|s}Q^{ij}_{\eta,s}\Phi^T_{j,s}\\
&\quad-(1-\bar{\phi}_j)G_{i,s|s}\breve{\bar{H}}_{i,s}\mathcal{P}^{(1)}_{ij,s|s-1}\breve{\bar{H}}^T_{j,s}\Phi^T_{j,s}-\bar{\alpha}_j\bar{\phi}_jG_{i,s|s}\breve{\bar{H}}_{i,s}\bar{\mathcal{P}}^{(1)T}_{ji,s|s|s-1}(\vec{H}_{j,s}-\vec{E}_j)^T\Phi^T_{j,s}\\
&\quad+G_{i,s|s}\Xi_{ij,s}L^T_{j,s|s}.
\end{aligned} \tag{37}$$

Proof. See Appendix A.5 ■

Next, a useful result used in the foregoing derivation is given.

Lemma 4. *The cross-covariance* $\bar{\mathcal{P}}^{(1)}_{ij,s+1|s+1|s}=\mathbb{E}\big\{\mathcal{X}^{(1)}_{i,s+1}\tilde{\mathcal{X}}^{(1)T}_{j,s+1|s}\big\}$ *can be given as below:*

$$\begin{aligned}
\bar{\mathcal{P}}^{(1)}_{ij,s+1|s+1|s} &= \bar{A}^{(1)}_{i,s}\bar{\mathcal{P}}^{(1)}_{ij,s|s|s-1}\bar{A}^{(1)T}_{j,s}+\mathfrak{B}_{ij1,s}-\bar{A}^{(1)}_{i,s}\bar{\mathcal{P}}^{(1)}_{ij,s|s|s-1}\breve{\bar{H}}^T_{j,s}G^T_{j,s+1|s}+\bar{B}^{(1)}_{i,s}Q^{ij}_{W,s}\bar{B}^{(1)T}_{j,s}\\
&\quad+\mathfrak{B}_{ij3,s}-\mathfrak{D}_{ij3,s}+\mathfrak{D}^T_{ji1,s}+\mathfrak{B}^T_{ji3,s}+\mathfrak{A}_{ij1,s}-\mathfrak{D}_{ij4,s}+(1-\bar{\phi}_i)D_{i,s}\big(\bar{\mathcal{P}}^{(1)}_{ij,s|s|s-1}\\
&\quad-\mathcal{P}^{(1)}_{ij,s|s-1}\big)\bar{A}^{(1)T}_{j,s}+\mathfrak{B}^T_{ji4,s}-(1-\bar{\phi}_i)D_{i,s}\big(\bar{\mathcal{P}}^{(1)}_{ij,s|s|s-1}-\mathcal{P}^{(1)}_{ij,s|s-1}\big)\breve{\bar{H}}^T_{j,s}G^T_{j,s+1|s},\quad s\geq 0;\\
\bar{\mathcal{P}}^{(1)}_{ij,0|0|-1} &= \mathrm{diag}\big\{[I_{n_1}\ 0]V^{-1}P_0(V^T)^{-1}[I_{n_1}\ 0]^T,\ 0\big\}.
\end{aligned} \tag{38}$$

Proof. See Appendix A.6 ■

Remark 5. From the definition of $\mathcal{X}^{(1)}_{i,s}=\big[x^T_{1,s}\quad \mathcal{Z}^T_{i,s-1}\big]^T$, it is clear that the LF $\hat{\mathcal{X}}^{(1)}_{i,s|s}$ has the computational order of magnitude $O((n_1+m_i)^3)$ according to Theorems 1 and 2. Furthermore, it can be easily seen that the LF $\hat{\mathcal{X}}^{(2)}_{i,s|s}$ with low computational cost can be calculated by the linear combination of $\hat{\mathcal{X}}^{(1)}_{i,s|s}$. Compared with the augmented filters presented in [31,47] with the computational order of magnitude $O((n_1+3m_i)^3)$, it is obviously deduced that the LFs designed in this paper possess lower complexity and the computational cost can be further reduced.

Remark 6. From the augmented state vectors $\mathcal{X}^{(1)}_{i,s}=\big[x^T_{1,s}\quad \mathcal{Z}^T_{i,s-1}\big]^T$ and $\mathcal{X}^{(2)}_{i,s}=\big[x^T_{2,s}\quad \mathcal{Z}^T_{i,s}\big]^T$, the subsystem states $x_{1,s}$ and $x_{2,s}$ can be given by $x_{1,s}=T_1\mathcal{X}^{(1)}_{i,s}$ and $x_{2,s}=T_2\mathcal{X}^{(2)}_{i,s}$, where $T_1=[I_{n_1}\ 0]$ and $T_2=[I_{n_2}\ 0]$. Accordingly, the LFs and estimation error covariance matrices for the reduced-order subsystems can be obtained by $\hat{x}^{(1)}_{i,s|s}=T_1\hat{\mathcal{X}}^{(1)}_{i,s|s}$, $\hat{x}^{(2)}_{i,s|s}=T_2\hat{\mathcal{X}}^{(2)}_{i,s|s}$,

$P^{(1)}_{ij,s|s} = T_1\mathcal{P}^{(1)}_{ij,s|s}T_1^T$, $P^{(2)}_{ij,s|s} = T_2\mathcal{P}^{(2)}_{ij,s|s}T_2^T$ and $P^{(1,2)}_{ij,s|s} = T_1\mathcal{P}^{(1,2)}_{ij,s|s}T_2^T$, especially when $i = j$, we define $P^{(1)}_{i,s|s} = P^{(1)}_{ii,s|s}$, $P^{(2)}_{i,s|s} = P^{(2)}_{ii,s|s}$ and $P^{(1,2)}_{i,s|s} = P^{(1,2)}_{ii,s|s}$. It is obvious to see that the LFs $\hat{x}_{i,s|s}$ and the corresponding covariance matrices $P_{i,s|s}$ for the original singular systems can be described as $\hat{x}_{i,s|s} = V\begin{bmatrix}\hat{x}^{(1)T}_{i,s|s} & \hat{x}^{(2)T}_{i,s|s}\end{bmatrix}^T$ and $P_{i,s|s} = V\begin{bmatrix}P^{(1)}_{i,s|s} & P^{(1,2)}_{i,s|s} \\ P^{(2,1)}_{i,s|s} & P^{(2)}_{i,s|s}\end{bmatrix}V^T$, respectively.

3.3.3. DFF weighted by matrices

Theorem 3. *For the reduced-order subsystems (5), (6) and (7) with multiple sensors, we have the reduced-order optimal fusion filters*

$$\hat{x}^{(k)}_{0,s|s} = \Lambda_{k,s}\hat{\aleph}_{k,s|s}, \quad k = 1, 2,$$

where $\Lambda_{k,s} = (e^T_{n_k}\Sigma^{-1}_{k,s|s}e_{n_k})^{-1}e^T_{n_k}\Sigma^{-1}_{k,s|s}$, $\hat{\aleph}_{k,s|s} = \begin{bmatrix}\hat{x}^{(k)T}_{1,s|s} & \hat{x}^{(k)T}_{2,s|s} & \cdots & \hat{x}^{(k)T}_{N,s|s}\end{bmatrix}^T$, $e_{n_k} = \begin{bmatrix}I_{n_k} & \cdots & I_{n_k}\end{bmatrix}^T$, *and* $\Sigma_{k,s|s} = \begin{bmatrix}P^{(k)}_{ij,s|s}\end{bmatrix}_{Nn_k\times Nn_k}$ *whose* $n_k \times n_k$ *sub-block in the* (i, j) *place is* $P^{(k)}_{ij,s|s}$*. The error covariance of* $\hat{x}^{(k)}_{0,s|s}$ *is computed by:*

$$P^{(k)}_{0,s|s} = (e^T_{n_k}\Sigma^{-1}_{k,s|s}e_{n_k})^{-1},$$

and we have $P^{(k)}_{0,s|s} \le P^{(k)}_{i,s|s}$ $(i = 1, 2, \ldots, N)$.

Proof. From the optimal fusion estimation algorithm presented in [20], the proof is complete readily. ■

3.4. DFF for the original systems

Theorem 4. *For the original multi-sensor singular systems (1), (2) and (3), the DFF has the following form:*

$$\hat{x}_{0,s|s} = V\begin{bmatrix}\hat{x}^{(1)T}_{0,s|s} & \hat{x}^{(2)T}_{0,s|s}\end{bmatrix}^T.$$

The covariance of $\hat{x}_{0,s|s}$ *is calculated by:*

$$P_{0,s|s} = V\begin{bmatrix}P^{(1)}_{0,s|s} & P^{(1,2)}_{0,s|s} \\ P^{(2,1)}_{0,s|s} & P^{(2)}_{0,s|s}\end{bmatrix}V^T,$$

where the filtering error covariance matrices $P^{(1,2)}_{0,s|s}$ *and* $P^{(2,1)}_{0,s|s}$ *between* $\tilde{x}^{(1)}_{0,s|s}$ *and* $\tilde{x}^{(2)}_{0,s|s}$ *are calculated by:*

$$P^{(1,2)}_{0,s|s} = (e^T_{n_1}\Sigma^{-1}_{1,s|s}e_{n_1})^{-1}e^T_{n_1}\Sigma^{-1}_{1,s|s}\Sigma_{12,s|s}\Sigma^{-1}_{2,s|s}e_{n_2}(e^T_{n_2}\Sigma^{-1}_{2,s|s}e_{n_2})^{-1},$$

where $P^{(1,2)}_{0,s|s} = P^{(2,1)T}_{0,s|s}$ *and* $\Sigma_{12,s|s} = \begin{bmatrix}P^{(1,2)}_{ij,s|s}\end{bmatrix}_{Nn_1\times Nn_2}$.

Proof. From the transformation $x_s = V\begin{bmatrix}x^T_{1,s} & x^T_{2,s}\end{bmatrix}^T$, we can prove it easily. ■

Remark 7. Generally, the MSIFE algorithms can be categorized into two types: the centralized fusion estimation algorithm [32] and the distributed fusion estimation algorithm [20]. Specifically, the centralized fusion filter is designed based on augmented measurements from all sensors. Meanwhile, the centralized fusion estimation algorithm has high estimation accuracy, however, it is worthwhile to note that the proposed algorithm also has expensive computational cost because of the utilization of the augmentation approach. Once there is a faulty sensor existed in the communication environment, the centralized fusion filter will appear failure. In order to improve the reliability and reduce computational cost of the centralized fusion filter, the DFF based on the distributed matrix-weighted fusion algorithm is constructed to perform the state estimation in this paper. Moreover, it should be noticed that the new DFF is developed based on the projection theory, and then the proposed DFF algorithm is optimal. In particular, it can be known that the DFF has the computational order of magnitude $O((Nn_1)^3)$ by virtue of Theorem 3. In addition, the DFF with a distributed parallel structure is more convenient to realize the fault detection and isolation, which has stronger robustness and higher flexibility.

The distributed optimal fusion filtering scheme for the singular systems is given by the following steps.

Algorithm 1: The recursive algorithm for distributed fusion filter of the singular systems

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Step 1. Set the initial values $\hat{\mathcal{X}}^{(1)}_{i,0|-1} = \left[\left([I_{n_1}\ 0]V^{-1}\pi_0\right)^T\ 0^T\right]^T$, $\mathcal{P}^{(1)}_{i,0|-1} = \mathcal{P}^{(1)}_{ij,0|-1} = \bar{\mathcal{P}}^{(1)}_{ij,0|0|-1} = \mathrm{diag}\{[I_{n_1}\ 0]V^{-1}P_0(V^T)^{-1}[I_{n_1}\ 0]^T,\ 0\}$, $q^{(1)}_{i,0} = q^{(1)}_{ij,0} = \mathrm{diag}\{[I_{n_1}\ 0]V^{-1}(\pi_0\pi_0^T + P_0)(V^T)^{-1}[I_{n_1}\ 0]^T,\ 0\}$, where $i,j = 1,2,\cdots,N$, $i \neq j$, $\varrho_0 = \pi_0\pi_0^T + P_0$.
Step 2. Compute the second moment ϱ_s by (8) and the innovation $\varepsilon_{i,s}$ by (17).
Step 3. Compute the innovation covariance matrices $\Xi_{i,s}$ and $\Xi_{ij,s}$ by (18) and (19), the gain matrices $G_{i,s|s}$ by (25), $G_{i,s+1|s}$ by (26), $L_{i,s|s}$ by (27), the second moments $q^{(1)}_{i,s}$ and $q^{(1)}_{ij,s}$ for augmented state $\mathcal{X}^{(1)}_{i,s}$ by (20) and (21).
Step 4. Compute the filtering error covariance matrices $\mathcal{P}^{(1)}_{i,s|s}$ by (30), $P^{(1)}_{i,s|s} = T_1\mathcal{P}^{(1)}_{i,s|s}T_1^T$, $\mathcal{P}^{(2)}_{i,s|s}$ by (34), $P^{(2)}_{i,s|s} = T_2\mathcal{P}^{(2)}_{i,s|s}T_2^T$, $\mathcal{P}^{(1)}_{ij,s|s}$ by (32), $P^{(1)}_{ij,s|s} = T_1\mathcal{P}^{(1)}_{ij,s|s}T_1^T$, $\mathcal{P}^{(2)}_{ij,s|s}$ by (35), $P^{(2)}_{ij,s|s} = T_2\mathcal{P}^{(2)}_{ij,s|s}T_2^T$, the prediction error covariance matrices $\mathcal{P}^{(1)}_{i,s+1|s}$ by (31), $\mathcal{P}^{(1)}_{ij,s+1|s}$ by (33), the correlation matrices $\mathcal{P}^{(1,2)}_{i,s|s}$ by (36), $P^{(1,2)}_{i,s|s} = T_1\mathcal{P}^{(1,2)}_{i,s|s}T_2^T$, $\mathcal{P}^{(1,2)}_{ij,s|s}$ by (37), and $P^{(1,2)}_{ij,s|s} = T_1\mathcal{P}^{(1,2)}_{ij,s|s}T_2^T$.
Step 5. Compute the LFs $\hat{\mathcal{X}}^{(1)}_{i,s|s}$ by (22), $\hat{x}^{(1)}_{i,s|s} = T_1\hat{\mathcal{X}}^{(1)}_{i,s|s}$, $\hat{\mathcal{X}}^{(2)}_{i,s|s}$ by (24), $\hat{x}^{(2)}_{i,s|s} = T_2\hat{\mathcal{X}}^{(2)}_{i,s|s}$, and the local predictor $\hat{\mathcal{X}}^{(1)}_{i,s+1|s}$ by (23).
Step 6. Compute the reduced-order fusion filters $\hat{x}^{(k)}_{0,s|s}$, $k = 1,2$, by Theorem 3.
Step 7. Compute the DFF $\hat{x}_{0,s|s}$ for the original singular systems by Theorem 4.
Step 8. Let $s = s+1$, return to step 2.

---

4. An illustrative example

Consider the singular systems with three sensors as (1) and (2), where the related parameters are set as:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ \Phi_s = \begin{bmatrix} 1 & -0.1 & -2 & 0.7 \\ 1 & 0 & 0 & 0.1 \\ 2.2 & -1.1 & -3.6 & 1 \\ 0.4\sin(2s) & -0.5 & 4 & 1 \end{bmatrix},\ \Upsilon_s = \begin{bmatrix} -0.1\sin(s) & 0.5 \\ 0.1 & 0.1 \\ 0.1 & -0.2 \\ -0.4 & 0 \end{bmatrix},$$

$$C_{1,s} = \begin{bmatrix} 0.8 & 1 & -1 & -0.2 \\ 0.5 & 0 & 2 & \sin(s) \end{bmatrix},\ C_{2,s} = \begin{bmatrix} 0.6 & 0.3 & 0 & 2 \\ 0.2 & 1 & 1 & 0.5 \end{bmatrix},\ C_{3,s} = \begin{bmatrix} 0.5\cos(s) & 1 & -0.6 & 2 \\ -0.1 & -0.4 & 0.4 & 1.1 \end{bmatrix}.$$

We choose the nonsingular matrices

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then, we have the following standard form:

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_{1,s+1} \\ x_{2,s+1} \end{bmatrix} = \begin{bmatrix} \Phi_{11,s} & \Phi_{12,s} \\ \Phi_{21,s} & \Phi_{22,s} \end{bmatrix}\begin{bmatrix} x_{1,s} \\ x_{2,s} \end{bmatrix} + \begin{bmatrix} \Upsilon_{1,s} \\ \Upsilon_{2,s} \end{bmatrix}\varpi_s,$$

$$z_{i,s} = \begin{bmatrix} C^{(1)}_{i,s} & C^{(2)}_{i,s} \end{bmatrix}\begin{bmatrix} x_{1,s} \\ x_{2,s} \end{bmatrix} + f_i(x_s, \xi_{i,s}) + \vartheta_{i,s}, \quad i = 1,2,3$$

where $x_{1,s} \in \mathbb{R}^2$ and $x_{2,s} \in \mathbb{R}^2$ denote the reduced-order state vectors, $z_{i,s} \in \mathbb{R}^2$ represents the sensor measurement output, ϖ_s is the process noise, and $\vartheta_{i,s}$ is the measurement noise.

The function $f_i(x_s, \xi_{i,s})$ is given as follows:

$$\begin{aligned} f_1(x_s, \xi_{1,s}) = &\begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix}\left[0.2\mathrm{sign}(\bar{x}_{1,s})\bar{x}_{1,s}\bar{\xi}^{(1)}_{1,s} + 0.3\mathrm{sign}(\bar{x}_{2,s})\bar{x}_{2,s}\bar{\xi}^{(1)}_{2,s} + 0.4\mathrm{sign}(\bar{x}_{3,s})\bar{x}_{3,s}\bar{\xi}^{(1)}_{3,s}\right. \\ &\left.+0.5\mathrm{sign}(\bar{x}_{4,s})\bar{x}_{4,s}\bar{\xi}^{(1)}_{4,s}\right], \\ f_2(x_s, \xi_{2,s}) = &\begin{bmatrix} 0.4 \\ 0.5 \end{bmatrix}\left[0.3\mathrm{sign}(\bar{x}_{1,s})\bar{x}_{1,s}\bar{\xi}^{(2)}_{1,s} + 0.4\mathrm{sign}(\bar{x}_{2,s})\bar{x}_{2,s}\bar{\xi}^{(2)}_{2,s} + 0.5\mathrm{sign}(\bar{x}_{3,s})\bar{x}_{3,s}\bar{\xi}^{(2)}_{3,s}\right. \\ &\left.+0.6\mathrm{sign}(\bar{x}_{4,s})\bar{x}_{4,s}\bar{\xi}^{(2)}_{4,s}\right], \\ f_3(x_s, \xi_{3,s}) = &\begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix}\left[0.1\mathrm{sign}(\bar{x}_{1,s})\bar{x}_{1,s}\bar{\xi}^{(3)}_{1,s} + 0.2\mathrm{sign}(\bar{x}_{2,s})\bar{x}_{2,s}\bar{\xi}^{(3)}_{3,s} + 0.4\mathrm{sign}(\bar{x}_{3,s})\bar{x}_{3,s}\bar{\xi}^{(3)}_{3,s}\right. \\ &\left.+0.6\mathrm{sign}(\bar{x}_{4,s})\bar{x}_{4,s}\bar{\xi}^{(3)}_{4,s}\right], \end{aligned}$$

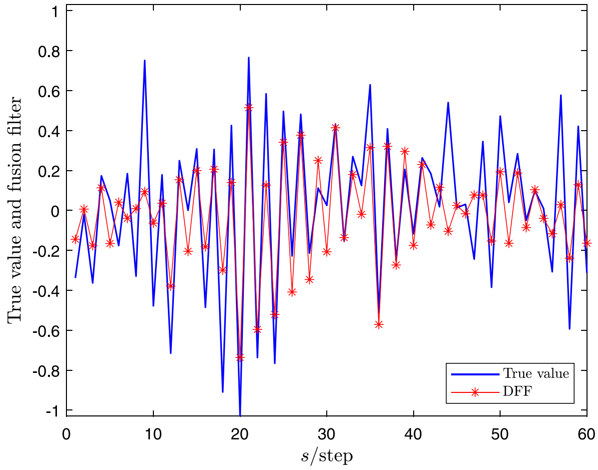

Fig. 1. The first component of the true state x_s and its fusion filter $\hat{x}_{0,s|s}$.

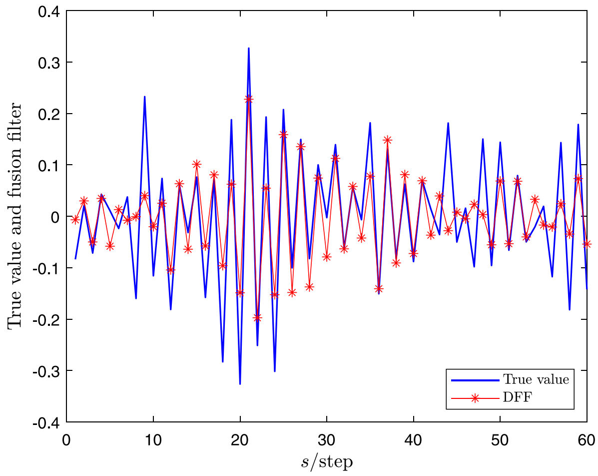

Fig. 2. The second component of the true state x_s and its fusion filter $\hat{x}_{0,s|s}$.

where $\bar{x}_{l,s}$ and $\bar{\xi}_{l,s}^{(i)}$ $(l = 1, 2, 3, 4; i = 1, 2, 3)$ represent the lth component of x_s and $\xi_{i,s}$, respectively. $\bar{\xi}_{l,s}^{(i)}$ is uncorrelated Gaussian white noise with zero mean and unity variance. We can easily get

$$\varPi_1 = \begin{bmatrix}0.5\\0.3\end{bmatrix}\begin{bmatrix}0.5\\0.3\end{bmatrix}^T, \varPi_2 = \begin{bmatrix}0.4\\0.5\end{bmatrix}\begin{bmatrix}0.4\\0.5\end{bmatrix}^T, \varPi_3 = \begin{bmatrix}0.3\\0.4\end{bmatrix}\begin{bmatrix}0.3\\0.4\end{bmatrix}^T,$$

$$\varGamma_1 = \text{diag}\{0.04, 0.09, 0.16, 0.25\}, \varGamma_2 = \text{diag}\{0.09, 0.16, 0.25, 0.36\}, \varGamma_3 = \text{diag}\{0.01, 0.04, 0.16, 0.36\}.$$

In the simulation, we simulate the collection of 60 data points and the initial value is a zero-mean Gaussian variable with $P_0 = I_4$. Other parameters are chosen as $Q_{\varpi,s} = 0.2I_2$, $Q_{\vartheta,s}^1 = 0.01I_2$, $Q_{\vartheta,s}^2 = 0.02I_2$, $Q_{\vartheta,s}^3 = 0.03I_2$, $\bar{\alpha}_1 = 0.5$, $\bar{\alpha}_2 = 0.6$, $\bar{\alpha}_3 = 0.4$, $\bar{\beta}_1 = 0.5$, $\bar{\beta}_2 = 0.4$ and $\bar{\beta}_3 = 0.6$. From the values of random variables $\alpha_{i,s}$ and $\beta_{i,s}$, it is not difficult to see that the packet on time receiving probabilities are $\text{Prob}\{\alpha_{1,s} = 1\} = 0.5$, $\text{Prob}\{\alpha_{2,s} = 1\} = 0.4$ and $\text{Prob}\{\alpha_{3,s} = 1\} = 0.6$, the one-step delay receiving probabilities are $\text{Prob}\{\alpha_{1,s} = 0, \alpha_{1,s-1} = 0, \beta_{1,s} = 1\} = 0.125$, $\text{Prob}\{\alpha_{2,s} = 0, \alpha_{2,s-1} = 0, \beta_{2,s} = 1\} = 0.064$ and $\text{Prob}\{\alpha_{3,s} = 0, \alpha_{3,s-1} = 0, \beta_{3,s} = 1\} = 0.216$, and the compensation probabilities are $\text{Prob}\{\alpha_{1,s} = 0, (1 - \alpha_{1,s-1})\beta_{1,s} = 0\} = 0.375$, $\text{Prob}\{\alpha_{2,s} = 0, (1 - \alpha_{2,s-1})\beta_{2,s} = 0\} = 0.336$ and $\text{Prob}\{\alpha_{3,s} = 0, (1 - \alpha_{3,s-1})\beta_{3,s} = 0\} = 0.384$. From Theorems 1, 3, and 4, the LFs $\hat{x}_{i,s|s}$ and DFF $\hat{x}_{0,s|s}$ can be obtained. Furthermore, we can obtain the cross-covariance matrices between any two sensor subsystems by Theorem 2. The simulation results are given in Figs. 1–10.

The proposed DFF is shown in Figs. 1–4, which can be seen that the state can be well estimated by the proposed DFF. To further show the performance of DFF algorithm, the mean square errors (MSEs) are plotted in Figs. 5–8 to evaluate the

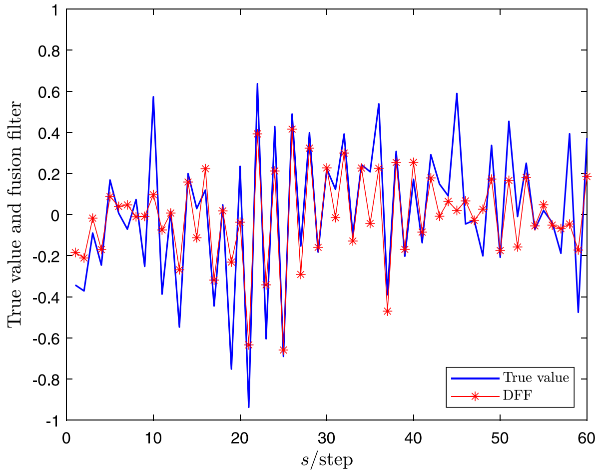

Fig. 3. The third component of the true state x_s and its fusion filter $\hat{x}_{0,s|s}$.

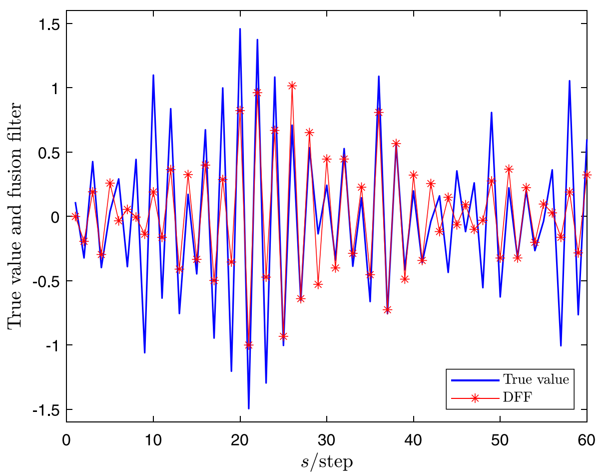

Fig. 4. The fourth component of the true state x_s and its fusion filter $\hat{x}_{0,s|s}$.

Table 1
The values of random variables.

	$\bar{\alpha}_1$	$\bar{\alpha}_2$	$\bar{\alpha}_3$	$\bar{\beta}_i(i=1,2,3)$
Case 1	0.2	0.5	0.6	0.5
Case 2	0.4	0.5	0.6	0.5
Case 3	0.4	0.5	0.7	0.5
	$\bar{\beta}_1$	$\bar{\beta}_2$	$\bar{\beta}_3$	$\bar{\alpha}_i(i=1,2,3)$
Case 4	0.2	0.5	0.6	0.5
Case 5	0.4	0.5	0.6	0.5
Case 6	0.4	0.5	0.7	0.5

algorithm performance, where the MSEs of 1000 times (i.e., $\frac{1}{1000}\sum_{l=1}^{1000}(x_s^l - \hat{x}_{i,s|s}^l)^2$ with $i = 0, 1, 2, 3$) are for LFs when $i = 1, 2, 3$ and DFF when $i = 0$. In Figs. 5–8, it is not difficult to see that the MSEs of the DFF are always below the LFs, which can be shown that the proposed fusion algorithm in this paper has better estimation performance. In particular, the comparison of traces of covariance matrices for DFF and LFs are exhibited in Fig. 9, which can be obviously observed that the DFF and LFs have the following accuracy relationship: $\text{tr}(P_{0,s|s}) < \text{tr}(P_{i,s|s})$. To sum up, it is easily derived that the presented filter with compensation strategy has better accuracy to estimate the system state.

In addition, in order to analyze the influences of the RTDs and PDs on the performance of the DFF. The concrete values of random variables are chosen in Table 1. In Fig. 10, the traces of the distributed fusion filtering error covariances are portrayed under different probabilities. Specifically, it can be concluded that the traces of the distributed fusion filtering

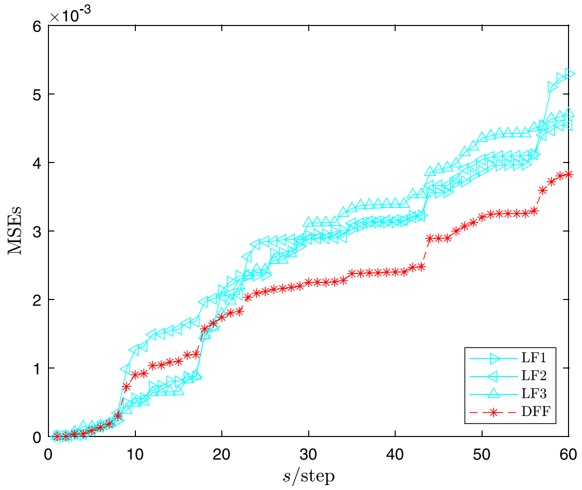

Fig. 5. MSEs of the first component of state x_s for LFs and DFF.

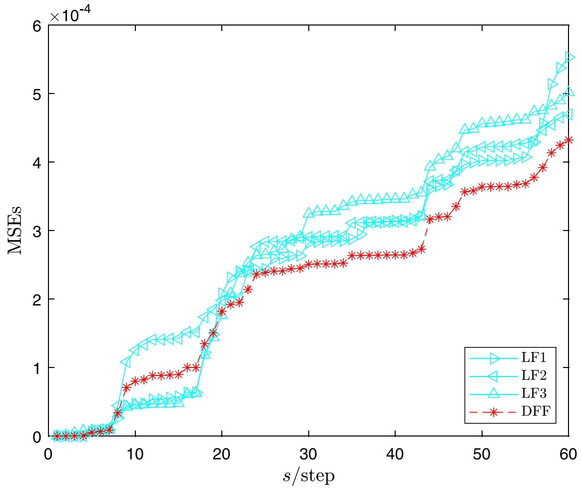

Fig. 6. MSEs of the second component of state x_s for LFs and DFF.

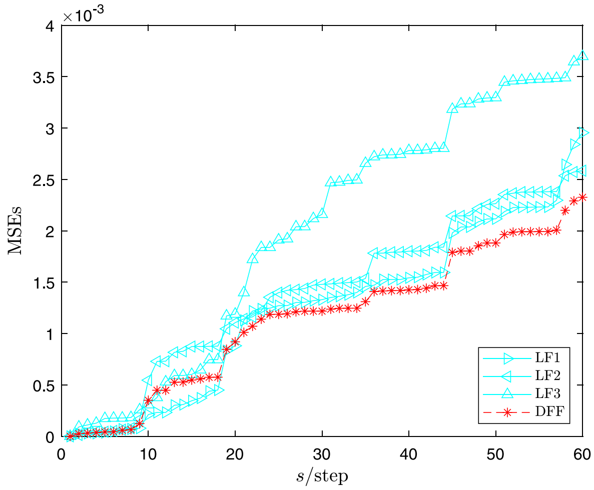

Fig. 7. MSEs of the third component of state x_s for LFs and DFF.

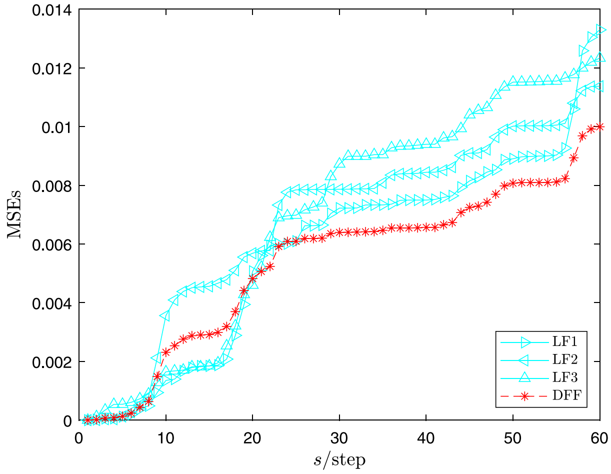

Fig. 8. MSEs of the fourth component of state x_s for LFs and DFF.

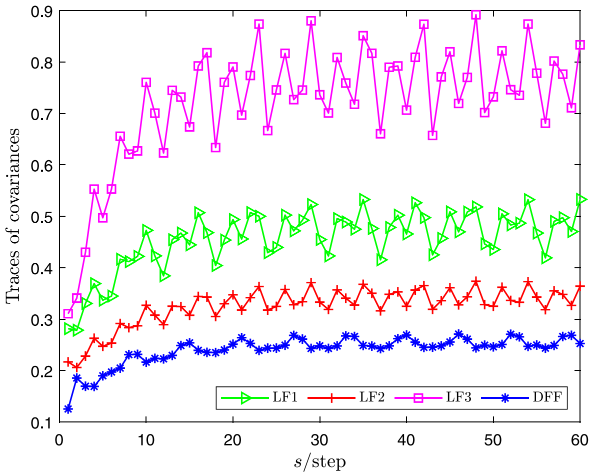

Fig. 9. Comparison of traces of covariance matrices for LFs and DFF.

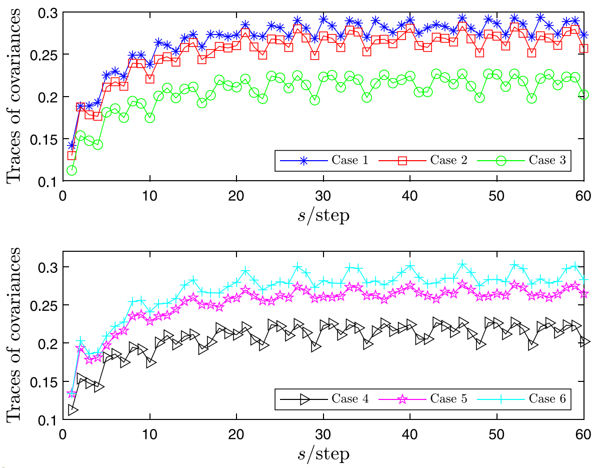

Fig. 10. Comparison of traces of covariance matrices for DFF with different values of $\bar{\alpha}_i$ $(i = 1, 2, 3)$ and $\bar{\beta}_i$ $(i = 1, 2, 3)$.

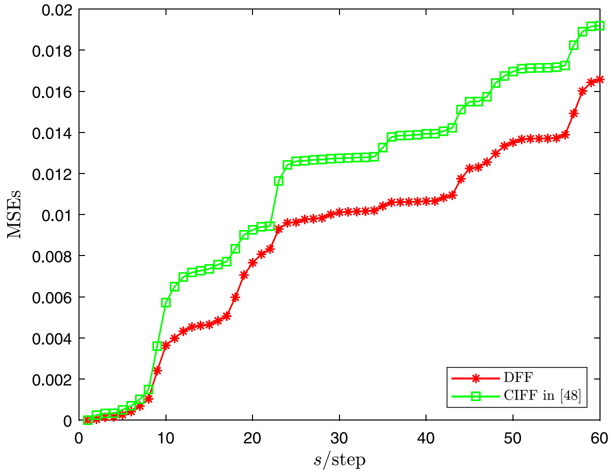

Fig. 11. Comparison of MSEs for DFF and CIFF in [48].

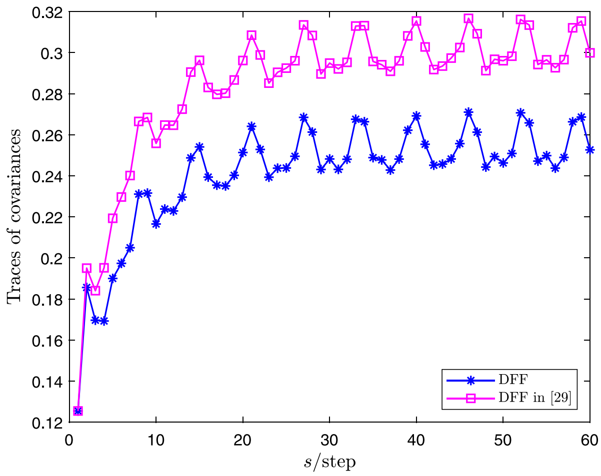

Fig. 12. Comparison of traces of covariance matrices for DFF and DFF in [29].

error covariances become smaller with increasing $\bar{\alpha}_i$ $(i = 1, 2, 3)$ and fixed $\bar{\beta}_i$ $(i = 1, 2, 3)$. On the other hand, when the value of $\bar{\alpha}_i$ $(i = 1, 2, 3)$ is fixed, the traces of the filtering error covariances become smaller as $\bar{\beta}_i$ $(i = 1, 2, 3)$ decreases. Hence, the proposed fusion filter performs better for higher on time receiving probability and lower one-step delay receiving probability.

To further verify the effectiveness of the proposed distributed fusion filtering algorithm, the comparison experiments are exhibited in Figs. 11–12. Specifically, the comparison of MSEs for DFF in this paper and covariance intersection fusion filter (CIFF) in [48] is given in Fig. 11, which can be seen that the estimation accuracy of the proposed DFF is better than CIFF. The main reason for this result is that the CIFF can effectively avoid computing the cross-covariance matrices. In addition, in order to demonstrate the superiority of the prediction compensation strategy in this paper, the accuracy comparison experiment is provided in Fig. 12 with respect to the prediction compensation strategy in this paper and zero-input compensation strategy in [29], which can be clearly observed that our proposed compensation strategy has better accuracy than [29] due to the fact that the prediction compensator uses all information received previously.

5. Conclusions

In this paper, we have investigated the MSIFE problem for TVSSs subject to RTDs and PD compensations. The singular systems have been converted into two nonsingular subsystems by employing the singular value decomposition method. Then, the LFs and corresponding estimation error covariances are derived based on new augmented systems by means of the projection theory. Subsequently, the matrix-weighted DFF has been presented for the original singular systems by utilizing the state transformation. In the simulation, we have shown that the DFF outperforms the LFs and we have analyzed the influences of RTDs and PDs on the DFF accuracy. Compared with the existing literature, the major features of the proposed DFF can be summarized as follows: (1) In contrast with the zero-input and hold-input schemes, the proposed prediction compensation strategy has better precision due to the fact that the historical measurement information has been used to compensate the impacts; (2) The LFs have been presented in the linear minimum variance sense, which has better accuracy than the suboptimal Kalman-type recursive filter; and (3) The DFF gives better performance on estimation accuracy compared with the existing covariance intersection fusion algorithm. In future work, we will extend our DFF algorithm to more complicated singular systems such as the systems with censored measurements, the state-saturated systems, and the networked systems under communication constraints.

CRediT authorship contribution statement

Jun Hu: Conceptualization, Methodology, Writing – original draft, Writing – review & editing, Investigation, Project administration, Funding acquisition. **Chen Wang:** Methodology, Writing – original draft, Investigation, Validation. **Raquel Caballero-Águila:** Writing – review. **Hongjian Liu:** Software, Validation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix

A.1. Proof of Lemma 1

Proof. Firstly, substituting $x_{s+1} = V\left[x_{1,s+1}^T \quad x_{2,s+1}^T\right]^T$ into $\varrho_{s+1} = \mathbb{E}\left\{x_{s+1}x_{s+1}^T\right\}$, we easily obtain (8). Then, substituting (5) into $\varrho_{1,s+1} = \mathbb{E}\left\{x_{1,s+1}x_{1,s+1}^T\right\}$ and using Assumptions 4–6 and $x_{1,s}\perp\varpi_s$, (9) can be proved as follows:

$$\begin{aligned}\varrho_{1,s+1} &= A_{1,s}\mathbb{E}\left\{x_{1,s}x_{1,s}^T\right\}A_{1,s}^T + B_{1,s}\mathbb{E}\left\{\varpi_s\varpi_s^T\right\}B_{1,s}^T + \left\{A_{1,s}\mathbb{E}\left\{x_{1,s}\varpi_s^T\right\}B_{1,s}^T\right\} + \left\{*\right\}^T \\ &= A_{1,s}\varrho_{1,s}A_{1,s}^T + B_{1,s}Q_{\varpi,s}B_{1,s}^T.\end{aligned}$$

Similarly, substituting (5) and (6) into $\varrho_{12,s+1} = \mathbb{E}\left\{x_{1,s+1}x_{2,s+1}^T\right\}$ and substituting (6) into $\varrho_{2,s+1} = \mathbb{E}\left\{x_{2,s+1}x_{2,s+1}^T\right\}$, we obtain (10) and (11). ■

A.2. Proof of Lemma 2

Proof. According to the projection theory, we get

$$\begin{aligned}\varepsilon_{i,s} &= \mathcal{Y}_{i,s} - \hat{\mathcal{Y}}_{i,s|s-1} \\ &= \breve{H}_{i,s}\mathcal{X}_{i,s}^{(1)} + \alpha_{i,s}f_i(x_s,\xi_{i,s}) + \alpha_{i,s}\eta_{i,s} - \bar{\breve{H}}_{i,s}\hat{\mathcal{X}}_{i,s|s-1}^{(1)} \\ &= (\alpha_{i,s}-\bar{\alpha}_i)(\vec{H}_{i,s}-\vec{E}_i)\mathcal{X}_{i,s}^{(1)} + \alpha_{i,s}f_i(x_s,\xi_{i,s}) + \alpha_{i,s}\eta_{i,s} + \bar{\breve{H}}_{i,s}\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}.\end{aligned} \tag{39}$$

Substituting (39) into $\Xi_{i,s} = \mathbb{E}\left\{\varepsilon_{i,s}\varepsilon_{i,s}^T\right\}$ yields

$$\begin{aligned}\Xi_{i,s} = {} & \mathbb{E}\left\{(\alpha_{i,s}-\bar{\alpha}_i)(\vec{H}_{i,s}-\vec{E}_i)\mathcal{X}_{i,s}^{(1)}\mathcal{X}_{i,s}^{(1)T}(\vec{H}_{i,s}-\vec{E}_i)^T(\alpha_{i,s}-\bar{\alpha}_i)\right\} + \mathbb{E}\left\{\alpha_{i,s}f_i(x_s,\xi_{i,s})f_i^T(x_s,\xi_{i,s})\alpha_{i,s}\right\} \\ & + \mathbb{E}\left\{\alpha_{i,s}\eta_{i,s}\eta_{i,s}^T\alpha_{i,s}\right\} + \mathbb{E}\left\{\bar{\breve{H}}_{i,s}\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}\tilde{\mathcal{X}}_{i,s|s-1}^{(1)T}\bar{\breve{H}}_{i,s}^T\right\} + \Big\{\mathbb{E}\left\{(\alpha_{i,s}-\bar{\alpha}_i)(\vec{H}_{i,s}-\vec{E}_i)\mathcal{X}_{i,s}^{(1)}f_i^T(x_s,\xi_{i,s})\alpha_{i,s}\right\} \\ & + \mathbb{E}\left\{(\alpha_{i,s}-\bar{\alpha}_i)(\vec{H}_{i,s}-\vec{E}_i)\mathcal{X}_{i,s}^{(1)}\eta_{i,s}^T\alpha_{i,s}\right\} + \mathbb{E}\left\{(\alpha_{i,s}-\bar{\alpha}_i)(\vec{H}_{i,s}-\vec{E}_i)\mathcal{X}_{i,s}^{(1)}\tilde{\mathcal{X}}_{i,s|s-1}^{(1)T}\bar{\breve{H}}_{i,s}^T\right\}\end{aligned}$$

$$+\mathbb{E}\Big\{\alpha_{i,s}f_i(x_s,\xi_{i,s})\eta_{i,s}^T\alpha_{i,s}\Big\}+\mathbb{E}\Big\{\alpha_{i,s}f_i(x_s,\xi_{i,s})\tilde{\mathcal{X}}_{i,s|s-1}^{(1)T}\bar{\bar{H}}_{i,s}^T\Big\}+\mathbb{E}\Big\{\alpha_{i,s}\eta_{i,s}\tilde{\mathcal{X}}_{i,s|s-1}^{(1)T}\bar{\bar{H}}_{i,s}^T\Big\}\Big\}+\Big\{*\Big\}^T,$$

for which, utilizing $\mathbb{E}\{\alpha_{i,s}-\bar{\alpha}_i\}=0$, Assumptions 4–6 and $\mathbb{E}\{\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}\eta_{i,s}^T\}=0$, we deduce (18). Similarly, (19) can be proved. ■

A.3. Proof of Lemma 3

Proof. Substituting (14) into $q_{i,s+1}^{(1)}=\mathbb{E}\{\mathcal{X}_{i,s+1}^{(1)}\mathcal{X}_{i,s+1}^{(1)T}\}$, we have

$$\begin{aligned}q_{i,s+1}^{(1)}=&\ \mathbb{E}\Big\{A_{i,s}^{(1)}\mathcal{X}_{i,s}^{(1)}\mathcal{X}_{i,s}^{(1)T}A_{i,s}^{(1)T}\Big\}+\mathbb{E}\Big\{B_{i,s}^{(1)}W_{i,s}W_{i,s}^TB_{i,s}^{(1)T}\Big\}+\mathbb{E}\Big\{\phi_{i,s}E_if_i(x_s,\xi_{i,s})f_i^T(x_s,\xi_{i,s})E_i^T\phi_{i,s}\Big\}\\&+\mathfrak{A}_{i1,s}+\mathbb{E}\Big\{(1-\phi_{i,s})D_{i,s}\hat{\mathcal{X}}_{i,s|s-1}^{(1)}\hat{\mathcal{X}}_{i,s|s-1}^{(1)T}D_{i,s}^T(1-\phi_{i,s})\Big\}+\Big\{\mathbb{E}\Big\{A_{i,s}^{(1)}\mathcal{X}_{i,s}^{(1)}W_{i,s}^TB_{i,s}^{(1)T}\Big\}\\&+\mathbb{E}\Big\{A_{i,s}^{(1)}\mathcal{X}_{i,s}^{(1)}f_i^T(x_s,\xi_{i,s})E_i^T\phi_{i,s}\Big\}+\mathfrak{B}_{i1,s}+\mathfrak{B}_{i2,s}+\mathbb{E}\Big\{B_{i,s}^{(1)}W_{i,s}f_i^T(x_s,\xi_{i,s})E_i^T\phi_{i,s}\Big\}+\mathfrak{B}_{i3,s}\\&+\mathbb{E}\Big\{B_{i,s}^{(1)}W_{i,s}\hat{\mathcal{X}}_{i,s|s-1}^{(1)T}D_{i,s}^T(1-\phi_{i,s})\Big\}+\mathbb{E}\Big\{\phi_{i,s}E_if_i(x_s,\xi_{i,s})\varepsilon_{i,s}^T\Phi_{i,s}^T(1-\phi_{i,s})\Big\}\\&+\mathbb{E}\Big\{\phi_{i,s}E_if_i(x_s,\xi_{i,s})\hat{\mathcal{X}}_{i,s|s-1}^{(1)T}D_{i,s}^T(1-\phi_{i,s})\Big\}+\mathfrak{B}_{i4,s}\Big\}+\Big\{*\Big\}^T.\end{aligned}\tag{40}$$

According to $\mathcal{X}_{i,s}^{(1)}\perp W_{i,s}$, Assumptions 4–6 and $\mathbb{E}\{\alpha_{i,s}\phi_{i,s}(1-\phi_{i,s})\}=0$, it is readily known that

$$\begin{aligned}&\mathbb{E}\Big\{A_{i,s}^{(1)}\mathcal{X}_{i,s}^{(1)}W_{i,s}^TB_{i,s}^{(1)T}\Big\}=0,\\&\mathbb{E}\Big\{A_{i,s}^{(1)}\mathcal{X}_{i,s}^{(1)}f_i^T(x_s,\xi_{i,s})E_i^T\phi_{i,s}\Big\}=0,\\&\mathbb{E}\Big\{B_{i,s}^{(1)}W_{i,s}f_i^T(x_s,\xi_{i,s})E_i^T\phi_{i,s}\Big\}=0,\\&\mathbb{E}\Big\{B_{i,s}^{(1)}W_{i,s}\hat{\mathcal{X}}_{i,s|s-1}^{(1)T}D_{i,s}^T(1-\phi_{i,s})\Big\}=0,\\&\mathbb{E}\Big\{\phi_{i,s}E_if_i(x_s,\xi_{i,s})\varepsilon_{i,s}^T\Phi_{i,s}^T(1-\phi_{i,s})\Big\}=0,\\&\mathbb{E}\Big\{\phi_{i,s}E_if_i(x_s,\xi_{i,s})\hat{\mathcal{X}}_{i,s|s-1}^{(1)T}D_{i,s}^T(1-\phi_{i,s})\Big\}=0.\end{aligned}$$

In addition, combining $A_{i,s}^{(1)}=\bar{A}_{1,s}+\phi_{i,s}\bar{H}_{i,s}$, $B_{i,s}^{(1)}=\bar{B}_{1,s}+\phi_{i,s}\bar{H}_{i,s}$, Assumptions 4–6, $\hat{\mathcal{X}}_{i,s|s-1}^{(1)}=\mathcal{X}_{i,s}^{(1)}-\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}$ and $\hat{\mathcal{X}}_{i,s|s-1}^{(1)}\perp\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}$ yield

$$\begin{aligned}&\mathbb{E}\Big\{A_{i,s}^{(1)}\mathcal{X}_{i,s}^{(1)}\mathcal{X}_{i,s}^{(1)T}A_{i,s}^{(1)T}\Big\}=\bar{A}_{1,s}q_{i,s}^{(1)}\bar{A}_{1,s}^T+\bar{\phi}_i\bar{H}_{i,s}q_{i,s}^{(1)}\bar{H}_{i,s}^T+\bar{\phi}_{i,s}\bar{A}_{1,s}q_{i,s}^{(1)}\bar{H}_{i,s}^T+\bar{\phi}_{i,s}\bar{H}_{i,s}q_{i,s}^{(1)}\bar{A}_{1,s}^T,\\&\mathbb{E}\Big\{B_{i,s}^{(1)}W_{i,s}W_{i,s}^TB_{i,s}^{(1)T}\Big\}=\bar{B}_{1,s}Q_{W,s}^i\bar{B}_{1,s}^T+\bar{\phi}_i\bar{E}_iQ_{W,s}^i\bar{E}_i^T+\bar{\phi}_i\bar{B}_{1,s}Q_{W,s}^i\bar{E}_i^T+\bar{\phi}_i\bar{E}_iQ_{W,s}^i\bar{B}_{1,s}^T,\\&\mathbb{E}\Big\{\phi_{i,s}E_if_i(x_s,\xi_{i,s})f_i^T(x_s,\xi_{i,s})E_i^T\phi_{i,s}\Big\}=\bar{\phi}_iE_i\sum_{l=1}^m\Pi_i^{(l)}\mathrm{tr}(\varrho_s\Gamma_i^{(l)})E_i^T,\\&\mathbb{E}\Big\{(1-\phi_{i,s})D_{i,s}\hat{\mathcal{X}}_{i,s|s-1}^{(1)}\hat{\mathcal{X}}_{i,s|s-1}^{(1)T}D_{i,s}^T(1-\phi_{i,s})\Big\}=(1-\bar{\phi}_i)D_{i,s}\big(q_{i,s}^{(1)}-\mathcal{P}_{i,s|s-1}^{(1)}\big)D_{i,s}^T.\end{aligned}$$

Subsequently, we have

$$\begin{aligned}\mathfrak{A}_{i1,s}&=\mathbb{E}\Big\{(1-\phi_{i,s})\Phi_{i,s}\varepsilon_{i,s}\varepsilon_{i,s}^T\Phi_{i,s}^T(1-\phi_{i,s})\Big\},\\\mathfrak{B}_{i1,s}&=\mathbb{E}\Big\{A_{i,s}^{(1)}\mathcal{X}_{i,s}^{(1)}\varepsilon_{i,s}^T\Phi_{i,s}^T(1-\phi_{i,s})\Big\},\\\mathfrak{B}_{i2,s}&=\mathbb{E}\Big\{A_{i,s}^{(1)}\mathcal{X}_{i,s}^{(1)}\hat{\mathcal{X}}_{i,s|s-1}^{(1)T}D_{i,s}^T(1-\phi_{i,s})\Big\},\\\mathfrak{B}_{i3,s}&=\mathbb{E}\Big\{B_{i,s}^{(1)}W_{i,s}\varepsilon_{i,s}^T\Phi_{i,s}^T(1-\phi_{i,s})\Big\},\\\mathfrak{B}_{i4,s}&=\mathbb{E}\Big\{(1-\phi_{i,s})\Phi_{i,s}\varepsilon_{i,s}\hat{\mathcal{X}}_{i,s|s-1}^{(1)T}D_{i,s}^T(1-\phi_{i,s})\Big\}.\end{aligned}$$

Combining innovation sequence (17) and $\hat{\mathcal{X}}_{i,s|s-1}^{(1)}=\mathcal{X}_{i,s}^{(1)}-\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}$, it is not difficult to derive the terms $\mathfrak{A}_{i1,s}$ and $\mathfrak{B}_{ik,s}(k=1,2,3,4)$. Then, the relationship (20) is true. Similarly, substituting (14) into $q_{ij,s+1}^{(1)}=\mathbb{E}\{\mathcal{X}_{i,s+1}^{(1)}\mathcal{X}_{j,s+1}^{(1)T}\}$ yields the following equations:

$$\begin{aligned}&\mathbb{E}\Big\{\phi_{i,s}E_if_i(x_s,\xi_{i,s})f_j^T(x_s,\xi_{j,s})E_j^T\phi_{j,s}\Big\}=0,\\&\mathbb{E}\Big\{A_{i,s}^{(1)}\mathcal{X}_{i,s}^{(1)}W_{j,s}^TB_{j,s}^{(1)T}\Big\}=0,\\&\mathbb{E}\Big\{A_{i,s}^{(1)}\mathcal{X}_{i,s}^{(1)}f_j^T(x_s,\xi_{j,s})E_j^T\phi_{j,s}\Big\}=0,\\&\mathbb{E}\Big\{B_{i,s}^{(1)}W_{i,s}f_j^T(x_s,\xi_{j,s})E_j^T\phi_{j,s}\Big\}=0,\\&\mathbb{E}\Big\{B_{i,s}^{(1)}W_{i,s}\hat{\mathcal{X}}_{j,s|s-1}^{(1)T}D_{j,s}^T(1-\phi_{j,s})\Big\}=0,\end{aligned}$$

$$
\begin{aligned}
&\mathbb{E}\{\phi_{i,s}E_{i}f_{i}(x_{i},\xi_{i,s})\hat{\mathcal{X}}_{j,s|s-1}^{(1)T}D_{j,s}^{T}(1-\phi_{j,s})\}=0,\\
&\mathbb{E}\{\phi_{i,s}E_{i}f_{i}(x_{s},\xi_{i,s})\varepsilon_{j,s}^{T}\Phi_{j,s}^{T}(1-\phi_{j,s})\}=0,\\
&\mathbb{E}\{A_{i,s}^{(1)}\mathcal{X}_{i,s}^{(1)}\mathcal{X}_{j,s}^{(1)T}A_{j,s}^{(1)T}\}=\bar{A}_{i,s}^{(1)}q_{ij,s}^{(1)}\bar{A}_{j,s}^{(1)T},\\
&\mathbb{E}\{B_{i,s}^{(1)}W_{i,s}W_{j,s}^{T}B_{j,s}^{(1)T}\}=\bar{B}_{i,s}^{(1)}Q_{W,s}^{ij}\bar{B}_{j,s}^{(1)T},\\
&\mathbb{E}\{(1-\phi_{i,s})D_{i,s}\hat{\mathcal{X}}_{i,s|s-1}^{(1)}\hat{\mathcal{X}}_{j,s|s-1}^{(1)T}D_{j,s}^{T}(1-\phi_{j,s})\}\\
=\;&(1-\bar{\phi}_{i})(1-\bar{\phi}_{j})D_{i,s}\big(q_{ij,s}^{(1)}+\mathcal{P}_{ij,s|s-1}^{(1)}-\bar{\mathcal{P}}_{ij,s|s|s-1}^{(1)}-\bar{\mathcal{P}}_{ji,s|s|s-1}^{(1)T}\big)D_{j,s}^{T}.
\end{aligned}
$$

Further, we have

$$
\begin{aligned}
\mathfrak{A}_{ij1,s}&=\mathbb{E}\{(1-\phi_{i,s})\Phi_{i,s}\varepsilon_{i,s}\varepsilon_{j,s}^{T}\Phi_{j,s}^{T}(1-\phi_{j,s})\},\\
\mathfrak{B}_{ij1,s}&=\mathbb{E}\{A_{i,s}^{(1)}\mathcal{X}_{i,s}^{(1)}\varepsilon_{j,s}^{T}\Phi_{j,s}^{T}(1-\phi_{j,s})\},\\
\mathfrak{B}_{ij2,s}&=\mathbb{E}\{A_{i,s}^{(1)}\mathcal{X}_{i,s}^{(1)}\hat{\mathcal{X}}_{j,s|s-1}^{(1)T}D_{j,s}^{T}(1-\phi_{j,s})\},\\
\mathfrak{B}_{ij3,s}&=\mathbb{E}\{B_{i,s}^{(1)}W_{i,s}\varepsilon_{j,s}^{T}\Phi_{j,s}^{T}(1-\phi_{j,s})\},\\
\mathfrak{B}_{ij4,s}&=\mathbb{E}\{(1-\phi_{i,s})\Phi_{i,s}\varepsilon_{i,s}\hat{\mathcal{X}}_{j,s|s-1}^{(1)T}D_{j,s}^{T}(1-\phi_{j,s})\}.
\end{aligned}
$$

Consequently, it can be concluded that (21) is true, and this proof is complete. ■

A.4. Proof of Theorem 1

Proof. Taking projection of both sides of (14) and (15) yields (22), (23) and (24) directly, where the gain matrices $G_{i,s|s}$, $G_{i,s+1|s}$ and $L_{i,s|s}$ are computed as follows:

$$
\begin{aligned}
G_{i,s|s}&=\mathbb{E}\{\mathcal{X}_{i,s}^{(1)}\varepsilon_{i,s}^{T}\}\Xi_{i,s}^{-1}, &(41)\\
G_{i,s+1|s}&=\mathbb{E}\{\mathcal{X}_{i,s+1}^{(1)}\varepsilon_{i,s}^{T}\}\Xi_{i,s}^{-1}, &(42)\\
L_{i,s|s}&=\mathbb{E}\{\mathcal{X}_{i,s}^{(2)}\varepsilon_{i,s}^{T}\}\Xi_{i,s}^{-1}. &(43)
\end{aligned}
$$

Subsequently, substituting (17) into (41), and using $\mathbb{E}\{\alpha_{i,s}-\bar{\alpha}_{i}\}=0$, $\hat{\mathcal{X}}_{i,s|s-1}^{(1)}\perp\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}$ and $\mathcal{X}_{i,s}^{(1)}\perp\eta_{i,s}$, we have

$$
\mathbb{E}\{\mathcal{X}_{i,s}^{(1)}\varepsilon_{i,s}^{T}\}=\mathcal{P}_{i,s|s-1}^{(1)}\bar{\bar{H}}_{i,s}^{T}.
$$

Then, utilizing the augmented state (14), we obtain

$$
\begin{aligned}
\mathbb{E}\{\mathcal{X}_{i,s+1}^{(1)}\varepsilon_{i,s}^{T}\}=\;&\mathbb{E}\{A_{i,s}^{(1)}\mathcal{X}_{i,s}^{(1)}\varepsilon_{i,s}^{T}\}+\mathbb{E}\{B_{i,s}^{(1)}W_{i,s}^{(1)}\varepsilon_{i,s}^{T}\}+\mathbb{E}\{\phi_{i,s}E_{i}f_{i}(x_{s},\xi_{i,s})\varepsilon_{i,s}^{T}\}\\
&+\mathbb{E}\{(1-\phi_{i,s})\Phi_{i,s}\varepsilon_{i,s}\varepsilon_{i,s}^{T}\}+\mathbb{E}\{(1-\phi_{i,s})D_{i,s}\hat{\mathcal{X}}_{i,s|s-1}^{(1)}\varepsilon_{i,s}^{T}\}.
\end{aligned}
$$

Based on the innovation sequence (17) and Remark 3, we have the following equations:

$$
\begin{aligned}
&\mathbb{E}\{A_{i,s}^{(1)}\mathcal{X}_{i,s}^{(1)}\varepsilon_{i,s}^{T}\}=\bar{A}_{i,s}^{(1)}\mathcal{P}_{i,s|s-1}^{(1)}\bar{\bar{H}}_{i,s}^{T}-\bar{\alpha}_{i}\bar{\phi}_{i}\bar{H}_{i,s}q_{i,s}^{(1)}(\vec{H}_{i,s}-\vec{E}_{i})^{T},\\
&\mathbb{E}\{B_{i,s}^{(1)}W_{i,s}^{(1)}\varepsilon_{i,s}^{T}\}=\bar{\alpha}_{i}\bar{B}_{1,s}S_{i,s},\\
&\mathbb{E}\{\phi_{i,s}E_{i}f_{i}(x_{s},\xi_{i,s})\varepsilon_{i,s}^{T}\}=0,\\
&\mathbb{E}\{(1-\phi_{i,s})D_{i,s}\hat{\mathcal{X}}_{i,s|s-1}^{(1)}\varepsilon_{i,s}^{T}\}=\bar{\alpha}_{i}\bar{\phi}_{i}D_{i,s}\big(q_{i,s}^{(1)}-\mathcal{P}_{i,s|s-1}^{(1)}\big)(\vec{H}_{i,s}-\vec{E}_{i})^{T}.
\end{aligned}
$$

Furthermore, we have

$$
\mathfrak{C}_{i1,s}=\mathbb{E}\{(1-\phi_{i,s})\Phi_{i,s}\varepsilon_{i,s}\varepsilon_{i,s}^{T}\},
$$

thus, it is not difficult to see that (26) is true. Similarly, substituting (17) into (43), we can prove that (27) is true. In addition, using (14) and $\mathbb{E}\{\varpi_{s}\varepsilon_{i,s}^{T}\}=\bar{\alpha}_{i}R_{i,s}$, we get

$$
K_{i,s+1|s}=\big(A_{1,s}\mathbb{E}\{x_{1,s}\varepsilon_{i,s}^{T}\}+\bar{\alpha}_{i}B_{1,s}R_{i,s}\big)\Xi_{i,s}^{-1},
$$

thus (29) can be derived. ■

A.5. Proof of Theorem 2

Proof. In order to prove it clearly, we take the following steps to prove this theorem.

Step 1: Let us derive the filtering error covariance $\mathcal{P}_{i,s|s}^{(1)}$ and the cross-covariance $\mathcal{P}_{ij,s|s}^{(1)}$. By subtracting (22) from $\mathcal{X}_{i,s}^{(1)}$ yields the filtering error equation

$$
\tilde{\mathcal{X}}_{i,s|s}^{(1)}=\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}-G_{i,s|s}\varepsilon_{i,s}. \tag{44}
$$

Next, substituting (44) into $\mathcal{P}^{(1)}_{i,s|s}=\mathbb{E}\left\{\tilde{\mathcal{X}}^{(1)}_{i,s|s}\tilde{\mathcal{X}}^{(1)T}_{i,s|s}\right\}$, we get

$$\mathcal{P}^{(1)}_{i,s|s}=\mathbb{E}\left\{\tilde{\mathcal{X}}^{(1)}_{i,s|s-1}\tilde{\mathcal{X}}^{(1)T}_{i,s|s-1}\right\}+\mathbb{E}\left\{G_{i,s|s}\varepsilon_{i,s}\varepsilon^T_{i,s}G^T_{i,s}\right\}-\left\{\mathbb{E}\left\{G_{i,s|s}\varepsilon_{i,s}\tilde{\mathcal{X}}^{(1)T}_{i,s|s-1}\right\}\right\}-\left\{*\right\}^T.$$

Combining $\hat{\mathcal{X}}^{(1)}_{i,s|s-1}\perp\varepsilon_{i,s}$ with $\mathbb{E}\left\{\mathcal{X}^{(1)}_{i,s}\varepsilon^T_{i,s}\right\}=G_{i,s|s}\Xi_{i,s}$, we have

$$\mathbb{E}\left\{\varepsilon_{i,s}\tilde{\mathcal{X}}^{(1)T}_{i,s|s-1}\right\}=\Xi_{i,s}G^T_{i,s|s},$$

then (30) can be proved easily. Similarly, based on innovation sequence (17), we have

$$\mathbb{E}\left\{\varepsilon_{i,s}\tilde{\mathcal{X}}^{(1)T}_{j,s|s-1}\right\}=\bar{\bar{H}}_{i,s}\mathcal{P}^{(1)}_{ij,s|s-1},$$

thus we can deduce (32).

Step 2: We are ready to provide the recisions of the prediction error covariance $\mathcal{P}^{(1)}_{i,s+1|s}$ and the cross-covariance $\mathcal{P}^{(1)}_{ij,s+1|s}$. From (23), the local prediction error equation for state $\mathcal{X}^{(1)}_{i,s}$ can be computed as follows:

$$\begin{aligned}\tilde{\mathcal{X}}^{(1)}_{i,s+1|s}=&A^{(1)}_{i,s}\tilde{\mathcal{X}}^{(1)}_{i,s|s-1}+B^{(1)}_{i,s}W_{i,s}+\phi_{i,s}E_if_i(x_s,\xi_{i,s})+(1-\phi_{i,s})\Phi_{i,s}\varepsilon_{i,s}\\&+(\phi_{i,s}-\bar{\phi}_i)(\bar{H}_{i,s}-D_{i,s})\hat{\mathcal{X}}^{(1)}_{i,s|s-1}-G_{i,s+1|s}\varepsilon_{i,s}.\end{aligned}\tag{45}$$

Substituting (45) into $\mathcal{P}^{(1)}_{i,s+1|s}=\mathbb{E}\left\{\tilde{\mathcal{X}}^{(1)}_{i,s+1|s}\tilde{\mathcal{X}}^{(1)T}_{i,s+1|s}\right\}$, we get

$$\begin{aligned}\mathcal{P}^{(1)}_{i,s+1|s}=&\mathbb{E}\left\{A^{(1)}_{i,s}\tilde{\mathcal{X}}^{(1)}_{i,s|s-1}\tilde{\mathcal{X}}^{(1)T}_{i,s|s-1}A^{(1)T}_{i,s}\right\}+\mathbb{E}\left\{B^{(1)}_{i,s}W_{i,s}W^T_{i,s}B^{(1)T}_{i,s}\right\}\\&+\mathbb{E}\left\{\phi_{i,s}E_if_i(x_s,\xi_{i,s})f^T_i(x_s,\xi_{i,s})E^T_i\phi_{i,s}\right\}+\mathbb{E}\left\{(1-\phi_{i,s})\Phi_{i,s}\varepsilon_{i,s}\varepsilon^T_{i,s}\Phi^T_{i,s}(1-\phi_{i,s})\right\}\\&+\mathbb{E}\left\{(\phi_{i,s}-\bar{\phi}_i)(\bar{H}_{i,s}-D_{i,s})\hat{\mathcal{X}}^{(1)}_{i,s|s-1}\hat{\mathcal{X}}^{(1)T}_{i,s|s-1}(\bar{H}_{i,s}-D_{i,s})^T(\phi_{i,s}-\bar{\phi}_i)\right\}\\&+\mathbb{E}\left\{G_{i,s+1|s}\varepsilon_{i,s}\varepsilon^T_{i,s}G^T_{i,s+1|s}\right\}+\left\{\mathbb{E}\left\{A^{(1)}_{i,s}\tilde{\mathcal{X}}^{(1)}_{i,s|s-1}W^T_{i,s}B^{(1)T}_{i,s}\right\}\right.\\&+\mathbb{E}\left\{A^{(1)}_{i,s}\tilde{\mathcal{X}}^{(1)}_{i,s|s-1}f^T_i(x_s,\xi_{i,s})E^T_i\phi_{i,s}\right\}+\mathbb{E}\left\{A^{(1)}_{i,s}\tilde{\mathcal{X}}^{(1)}_{i,s|s-1}\varepsilon^T_{i,s}\Phi^T_{i,s}(1-\phi_{i,s})\right\}\\&+\mathbb{E}\left\{A^{(1)}_{i,s}\tilde{\mathcal{X}}^{(1)}_{i,s|s-1}\hat{\mathcal{X}}^{(1)T}_{i,s|s-1}(\bar{H}_{i,s}-D_{i,s})^T(\phi_{i,s}-\bar{\phi}_i)\right\}\\&-\mathbb{E}\left\{A^{(1)}_{i,s}\tilde{\mathcal{X}}^{(1)}_{i,s|s-1}\varepsilon^T_{i,s}G^T_{i,s+1|s}\right\}+\mathbb{E}\left\{B^{(1)}_{i,s}W_{i,s}f^T_i(x_s,\xi_{i,s})E^T_i\phi_{i,s}\right\}\\&+\mathbb{E}\left\{B^{(1)}_{i,s}W_{i,s}\varepsilon^T_{i,s}\Phi^T_{i,s}(1-\phi_{i,s})\right\}+\mathbb{E}\left\{B^{(1)}_{i,s}W_{i,s}\hat{\mathcal{X}}^{(1)T}_{i,s|s-1}(\bar{H}_{i,s}-D_{i,s})^T(\phi_{i,s}-\bar{\phi}_i)\right\}\\&-\mathbb{E}\left\{B^{(1)}_{i,s}W_{i,s}\varepsilon^T_{i,s}G^T_{i,s+1|s}\right\}+\mathbb{E}\left\{\phi_{i,s}E_if_i(x_s,\xi_{i,s})\varepsilon^T_{i,s}\Phi^T_{i,s}(1-\phi_{i,s})\right\}\\&+\mathbb{E}\left\{\phi_{i,s}E_if_i(x_s,\xi_{i,s})\hat{\mathcal{X}}^{(1)T}_{i,s|s-1}(\bar{H}_{i,s}-D_{i,s})^T(\phi_{i,s}-\bar{\phi}_i)\right\}\\&-\mathbb{E}\left\{\phi_{i,s}E_if_i(x_s,\xi_{i,s})\varepsilon^T_{i,s}G^T_{i,s+1|s}\right\}-\mathbb{E}\left\{(1-\phi_{i,s})\Phi_{i,s}\varepsilon_{i,s}\varepsilon^T_{i,s}G^T_{i,s+1|s}\right\}\\&+\mathbb{E}\left\{(1-\phi_{i,s})\Phi_{i,s}\varepsilon_{i,s}\hat{\mathcal{X}}^{(1)T}_{i,s|s-1}(\bar{H}_{i,s}-D_{i,s})^T(\phi_{i,s}-\bar{\phi}_i)\right\}\\&\left.-\mathbb{E}\left\{(\phi_{i,s}-\bar{\phi}_i)(\bar{H}_{i,s}-D_{i,s})\hat{\mathcal{X}}^{(1)}_{i,s|s-1}\varepsilon^T_{i,s}G^T_{i,s+1|s}\right\}\right\}+\left\{*\right\}^T.\end{aligned}$$

It can be derived from $A^{(1)}_{i,s}=\bar{A}_{1,s}+\phi_{i,s}\bar{H}_{i,s}$ and $\hat{\mathcal{X}}^{(1)}_{i,s|s-1}=\mathcal{X}^{(1)}_{i,s}-\tilde{\mathcal{X}}^{(1)}_{i,s|s-1}$ that

$$\begin{aligned}&\mathbb{E}\left\{A^{(1)}_{i,s}\tilde{\mathcal{X}}^{(1)}_{i,s|s-1}\tilde{\mathcal{X}}^{(1)T}_{i,s|s-1}A^{(1)T}_{i,s}\right\}\\=&\bar{A}_{1,s}\mathcal{P}^{(1)}_{i,s|s-1}\bar{A}^T_{1,s}+\bar{\phi}_i\bar{H}_{i,s}\mathcal{P}^{(1)}_{i,s|s-1}\bar{H}^T_{i,s}+\left\{\bar{\phi}_i\bar{A}_{1,s}\mathcal{P}^{(1)}_{i,s|s-1}\bar{H}^T_{i,s}\right\}+\left\{*\right\}^T,\\&\mathbb{E}\left\{(\phi_{i,s}-\bar{\phi}_i)(\bar{H}_{i,s}-D_{i,s})\hat{\mathcal{X}}^{(1)}_{i,s|s-1}\hat{\mathcal{X}}^{(1)T}_{i,s|s-1}(\bar{H}_{i,s}-D_{i,s})^T(\phi_{i,s}-\bar{\phi}_i)\right\}\\=&(\bar{\phi}_i-\bar{\phi}^2_i)(\bar{H}_{i,s}-D_{i,s})\left(q^{(1)}_{i,s}-\mathcal{P}^{(1)}_{i,s|s-1}\right)(\bar{H}_{i,s}-D_{i,s})^T,\\&\mathbb{E}\left\{G_{i,s+1|s}\varepsilon_{i,s}\varepsilon^T_{i,s}G^T_{i,s+1|s}\right\}=G_{i,s+1|s}\Xi_{i,s}G^T_{i,s+1|s}.\end{aligned}$$

Besides, based on Assumptions 4–6, we have the following equations

$$\begin{aligned}&\mathbb{E}\left\{A^{(1)}_{i,s}\tilde{\mathcal{X}}^{(1)}_{i,s|s-1}W^T_{i,s}B^{(1)T}_{i,s}\right\}=0,\\&\mathbb{E}\left\{A^{(1)}_{i,s}\tilde{\mathcal{X}}^{(1)}_{i,s|s-1}f^T_i(x_s,\xi_{i,s})E^T_i\phi_{i,s}\right\}=0,\\&\mathbb{E}\left\{A^{(1)}_{i,s}\tilde{\mathcal{X}}^{(1)}_{i,s|s-1}\hat{\mathcal{X}}^{(1)T}_{i,s|s-1}(\bar{H}_{i,s}-D_{i,s})^T(\phi_{i,s}-\bar{\phi}_i)\right\}=0,\\&\mathbb{E}\left\{B^{(1)}_{i,s}W_{i,s}f^T_i(x_s,\xi_{i,s})E^T_i\phi_{i,s}\right\}=0,\\&\mathbb{E}\left\{B^{(1)}_{i,s}W_{i,s}\hat{\mathcal{X}}^{(1)T}_{i,s|s-1}(\bar{H}_{i,s}-D_{i,s})^T(\phi_{i,s}-\bar{\phi}_i)\right\}=0,\end{aligned}$$

$$\mathbb{E}\{\phi_{i,s}E_if_i(x_s,\xi_{i,s})\varepsilon_{i,s}^T\Phi_{i,s}^T(1-\phi_{i,s})\}=0,$$
$$\mathbb{E}\{\phi_{i,s}E_if_i(x_s,\xi_{i,s})\varepsilon_{i,s}^TG_{i,s+1|s}^T\}=0.$$

In addition, we have

$$\begin{aligned}
\mathfrak{D}_{i1,s}&=\mathbb{E}\{A_{i,s}^{(1)}\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}\varepsilon_{i,s}^T\Phi_{i,s}^T(1-\phi_{i,s})\},\\
\mathfrak{D}_{i2,s}&=\mathbb{E}\{A_{i,s}^{(1)}\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}\varepsilon_{i,s}^TG_{i,s+1|s}^T\},\\
\mathfrak{D}_{i3,s}&=\mathbb{E}\{B_{i,s}^{(1)}W_{i,s}\varepsilon_{i,s}^TG_{i,s+1|s}^T\},\\
\mathfrak{D}_{i4,s}&=\mathbb{E}\{(1-\phi_{i,s})\Phi_{i,s}\varepsilon_{i,s}\hat{\mathcal{X}}_{i,s|s-1}^{(1)T}(\bar{H}_{i,s}-D_{i,s})^T(\phi_{i,s}-\bar{\phi}_i)\},\\
\mathfrak{D}_{i5,s}&=\mathbb{E}\{(1-\phi_{i,s})\Phi_{i,s}\varepsilon_{i,s}\varepsilon_{i,s}^TG_{i,s+1|s}^T\},\\
\mathfrak{D}_{i6,s}&=\mathbb{E}\{(\phi_{i,s}-\bar{\phi}_i)(\bar{H}_{i,s}-D_{i,s})\hat{\mathcal{X}}_{i,s|s-1}^{(1)}\varepsilon_{i,s}^TG_{i,s+1|s}^T\}.
\end{aligned}$$

Accordingly, it is easy to derive the terms $\mathfrak{D}_{ik,s}$ $(k=1,2,\dots,6)$. Then, the relationship (31) is true.

Substituting (45) into $\mathcal{P}_{ij,s+1|s}^{(1)}=\mathbb{E}\{\tilde{\mathcal{X}}_{i,s+1|s}^{(1)}\tilde{\mathcal{X}}_{j,s+1|s}^{(1)T}\}$, it is not difficult to obtain that the following terms are true:

$$\mathbb{E}\{A_{i,s}^{(1)}\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}\tilde{\mathcal{X}}_{j,s|s-1}^{(1)T}A_{j,s}^{(1)T}\}=\bar{A}_{i,s}^{(1)}\mathcal{P}_{ij,s|s-1}^{(1)}\bar{A}_{j,s}^{(1)T},$$
$$\mathbb{E}\{B_{i,s}^{(1)}W_{i,s}W_{j,s}^TB_{j,s}^{(1)T}\}=\bar{B}_{i,s}^{(1)}Q_{W,s}^{ij}\bar{B}_{j,s}^{(1)T},$$
$$\mathbb{E}\{G_{i,s+1|s}\varepsilon_{i,s}\varepsilon_{j,s}^TG_{j,s+1|s}^T\}=G_{i,s+1|s}\Xi_{ij,s}G_{j,s+1|s}^T.$$

Further, we have

$$\begin{aligned}
\mathfrak{D}_{ij1,s}&=\mathbb{E}\{A_{i,s}^{(1)}\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}\varepsilon_{j,s}^T\Phi_{j,s}^T(1-\phi_{j,s})\},\quad \mathfrak{D}_{ij2,s}=\mathbb{E}\{A_{i,s}^{(1)}\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}\varepsilon_{j,s}^TG_{j,s+1|s}^T\},\\
\mathfrak{D}_{ij3,s}&=\mathbb{E}\{B_{i,s}^{(1)}W_{i,s}\varepsilon_{j,s}^TG_{j,s+1|s}^T\},\quad \mathfrak{D}_{ij4,s}=\mathbb{E}\{(1-\phi_{i,s})\Phi_{i,s}\varepsilon_{i,s}\varepsilon_{j,s}^TG_{j,s+1|s}^T\},
\end{aligned}$$

from which we can easily obtain that the terms $\mathfrak{D}_{ijk,s}$ $(k=1,2,3,4)$ are true. Then, we can prove (33).

Step 3: Let us present the recursions of the filtering error covariance $\mathcal{P}_{i,s|s}^{(2)}$ and the cross-covariance $\mathcal{P}_{ij,s|s}^{(2)}$. Subtracting (24) from $\mathcal{X}_{i,s}^{(2)}$, we get the filtering error for $\mathcal{X}_{i,s}^{(2)}$ as follows:

$$\begin{aligned}
\tilde{\mathcal{X}}_{i,s|s}^{(2)}&=A_{i,s}^{(1)}\tilde{\mathcal{X}}_{i,s|s-1}^{(2)}+B_{i,s}^{(2)}W_{i,s}+\phi_{i,s}E_if_i(x_s,\xi_{i,s})+(1-\phi_{i,s})\Phi_{i,s}\varepsilon_{i,s}\\
&\quad+(\phi_{i,s}-\bar{\phi}_i)(\bar{H}_{i,s}-D_{i,s})\hat{\mathcal{X}}_{i,s|s-1}^{(1)}-L_{i,s|s}\varepsilon_{i,s},
\end{aligned}\tag{46}$$

thus (34) and (35) can be proved.

Step 4: We are ready to derive the correlation matrices $\mathcal{P}_{i,s|s}^{(1,2)}$ and $\mathcal{P}_{ij,s|s}^{(1,2)}$. From (44) and (46), we can obtain $\mathcal{P}_{i,s|s}^{(1,2)}=\mathbb{E}\{\tilde{\mathcal{X}}_{i,s|s}^{(1)}\tilde{\mathcal{X}}_{i,s|s}^{(2)T}\}$ as follows:

$$\begin{aligned}
\mathcal{P}_{i,s|s}^{(1,2)}&=\mathbb{E}\{\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}\tilde{\mathcal{X}}_{i,s|s-1}^{(1)T}A_{i,s}^{(2)T}\}+\mathbb{E}\{\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}W_{i,s}^TB_{i,s}^{(2)T}\}+\mathbb{E}\{\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}f_i^T(x_s,\xi_{i,s})E_i^T\phi_{i,s}\}\\
&\quad+\mathbb{E}\{\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}\varepsilon_{i,s}^T\Phi_{i,s}^T(1-\phi_{i,s})\}+\mathbb{E}\{\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}\hat{\mathcal{X}}_{i,s|s-1}^{(1)T}(\bar{H}_{i,s}-D_{i,s})^T(\phi_{i,s}-\bar{\phi}_i)\}\\
&\quad-\mathbb{E}\{\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}\varepsilon_{i,s}^TL_{i,s|s}^T\}-\mathbb{E}\{G_{i,s|s}\varepsilon_{i,s}\tilde{\mathcal{X}}_{i,s|s-1}^{(1)T}A_{i,s}^{(2)T}\}-\mathbb{E}\{G_{i,s|s}\varepsilon_{i,s}W_{i,s}^TB_{i,s}^{(2)T}\}\\
&\quad-\mathbb{E}\{G_{i,s|s}\varepsilon_{i,s}f_i^T(x_s,\xi_{i,s})E_i^T\phi_{i,s}\}-\mathbb{E}\{G_{i,s|s}\varepsilon_{i,s}\varepsilon_{i,s}^T\Phi_{i,s}^T(1-\phi_{i,s})\}\\
&\quad-\mathbb{E}\{G_{i,s|s}\varepsilon_{i,s}\hat{\mathcal{X}}_{i,s|s-1}^{(1)T}(\bar{H}_{i,s}-D_{i,s})^T(\phi_{i,s}-\bar{\phi}_i)\}+\mathbb{E}\{G_{i,s|s}\varepsilon_{i,s}\varepsilon_{i,s}^TL_{i,s|s}^T\}.
\end{aligned}$$

Then, in light of Assumptions 4–6, we have the following zero-terms:

$$\mathbb{E}\{\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}W_{i,s}^TB_{i,s}^{(2)T}\}=0,\quad \mathbb{E}\{\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}f_i^T(x_s,\xi_{i,s})E_i^T\phi_{i,s}\}=0,$$
$$\mathbb{E}\{\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}\hat{\mathcal{X}}_{i,s|s-1}^{(1)T}(\bar{H}_{i,s}-D_{i,s})^T(\phi_{i,s}-\bar{\phi}_i)\}=0,\quad \mathbb{E}\{G_{i,s|s}\varepsilon_{i,s}f_i^T(x_s,\xi_{i,s})E_i^T\phi_{i,s}\}=0.$$

Further, we have

$$\begin{aligned}
&\mathbb{E}\{\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}\tilde{\mathcal{X}}_{i,s|s-1}^{(1)T}A_{i,s}^{(2)T}\}=\mathcal{P}_{i,s|s-1}^{(1)}\bar{A}_{i,s}^{(2)},\\
&\mathbb{E}\{\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}\varepsilon_{i,s}^T\Phi_{i,s}^T(1-\phi_{i,s})\}=\bar{\alpha}_i\bar{\phi}_i\mathcal{P}_{i,s|s-1}^{(1)}(\vec{H}_{i,s}-\vec{E}_i)^T\Phi_{i,s}^T+(1-\bar{\phi}_i)\mathcal{P}_{i,s|s-1}^{(1)}\breve{\bar{H}}_{i,s}^T\Phi_{i,s}^T,\\
&\mathbb{E}\{\tilde{\mathcal{X}}_{i,s|s-1}^{(1)}\varepsilon_{i,s}^TL_{i,s|s}^T\}=\mathcal{P}_{i,s|s-1}^{(1)}\breve{\bar{H}}_{i,s}^TL_{i,s|s}^T,\\
&\mathbb{E}\{G_{i,s|s}\varepsilon_{i,s}\tilde{\mathcal{X}}_{i,s|s-1}^{(1)T}A_{i,s}^{(2)T}\}=G_{i,s|s}\breve{\bar{H}}_{i,s}\mathcal{P}_{i,s|s-1}^{(1)}\bar{A}_{i,s}^{(2)T}-\bar{\alpha}_i\bar{\phi}_iG_{i,s|s}(\vec{H}_{i,s}-\vec{E}_i)\mathcal{P}_{i,s|s-1}^{(1)}\bar{H}_{i,s}^T,\\
&\mathbb{E}\{G_{i,s|s}\varepsilon_{i,s}W_{i,s}^TB_{i,s}^{(2)T}\}=\bar{\alpha}_iG_{i,s|s}S_{i,s}^T\bar{B}_{2,s}^T,\\
&\mathbb{E}\{G_{i,s|s}\varepsilon_{i,s}\hat{\mathcal{X}}_{i,s|s-1}^{(1)T}(\bar{H}_{i,s}-D_{i,s})^T(\phi_{i,s}-\bar{\phi}_i)\}=\bar{\alpha}_i\bar{\phi}_iG_{i,s|s}(\vec{H}_{i,s}-\vec{E}_i)(\mathcal{P}_{i,s|s-1}^{(1)}-q_{i,s}^{(1)})(\bar{H}_{i,s}-D_{i,s})^T,\\
&\mathbb{E}\{G_{i,s|s}\varepsilon_{i,s}\varepsilon_{i,s}^TL_{i,s|s}^T\}=G_{i,s|s}\Xi_{i,s}L_{i,s|s}^T.
\end{aligned}$$

Consequently, it is readily known that the relationship (36) holds.

Similarly, substituting (44) and (46) into $\mathcal{P}^{(1,2)}_{ij,s|s} = \mathbb{E}\{\tilde{\mathcal{X}}^{(1)}_{i,s|s}\tilde{\mathcal{X}}^{(2)T}_{j,s|s}\}$, based on Assumptions 4–6, we have

$$\mathbb{E}\{\tilde{\mathcal{X}}^{(1)}_{i,s|s-1}W^T_{j,s}B^{(2)T}_{j,s}\} = 0,$$
$$\mathbb{E}\{\tilde{\mathcal{X}}^{(1)}_{i,s|s-1}f^T_j(x_s,\xi_{j,s})E^T_j\phi_{j,s}\} = 0,$$
$$\mathbb{E}\{\tilde{\mathcal{X}}^{(1)}_{i,s|s-1}\hat{\mathcal{X}}^{(1)T}_{j,s|s-1}(\bar{H}_{j,s}-D_{j,s})^T(\phi_{j,s}-\bar{\phi}_j)\} = 0,$$
$$\mathbb{E}\{G_{i,s|s}\varepsilon_{i,s}f^T_j(x_s,\xi_{j,s})E^T_j\phi_{j,s}\} = 0,$$
$$\mathbb{E}\{G_{i,s|s}\varepsilon_{i,s}\hat{\mathcal{X}}^{(1)T}_{j,s|s-1}(\bar{H}_{j,s}-D_{j,s})^T(\phi_{j,s}-\bar{\phi}_j)\} = 0.$$

Further, we get

$$\mathbb{E}\{\tilde{\mathcal{X}}^{(1)}_{i,s|s-1}\tilde{\mathcal{X}}^{(1)T}_{j,s|s-1}A^{(2)T}_{j,s}\} = \mathcal{P}^{(1)}_{ij,s|s-1}\bar{A}^{(2)}_{j,s},$$
$$\mathbb{E}\{\tilde{\mathcal{X}}^{(1)}_{i,s|s-1}\varepsilon^T_{j,s}\Phi^T_{j,s}(1-\phi_{j,s})\} = \bar{\alpha}_j\bar{\phi}_j\bar{\mathcal{P}}^{(1)T}_{ji,s|s|s-1}(\vec{H}_{j,s}-\vec{E}_j)^T\Phi^T_{j,s} + (1-\bar{\phi}_j)\mathcal{P}^{(1)}_{ij,s|s-1}\breve{\bar{H}}^T_{j,s}\Phi^T_{j,s},$$
$$\mathbb{E}\{\tilde{\mathcal{X}}^{(1)}_{i,s|s-1}\varepsilon^T_{j,s}L^T_{j,s|s}\} = \mathcal{P}^{(1)}_{ij,s|s-1}\breve{\bar{H}}^T_{j,s}L^T_{j,s|s},$$
$$\mathbb{E}\{G_{i,s|s}\varepsilon_{i,s}\tilde{\mathcal{X}}^{(1)T}_{j,s|s-1}A^{(2)T}_{j,s}\} = G_{i,s|s}\breve{\bar{H}}_{i,s}\mathcal{P}^{(1)}_{ij,s|s-1}\bar{A}^{(2)T}_{j,s},$$
$$\mathbb{E}\{G_{i,s|s}\varepsilon_{i,s}W^T_{j,s}B^{(2)T}_{j,s}\} = \bar{\alpha}_iG_{i,s|s}S^T_{ji,s}\bar{B}^{(2)T}_{j,s},$$
$$\begin{aligned}\mathbb{E}\{G_{i,s|s}\varepsilon_{i,s}\varepsilon^T_{j,s}\Phi^T_{j,s}(1-\phi_{j,s})\} = {}& \bar{\alpha}_i\bar{\alpha}_jG_{i,s|s}Q^{ij}_{\eta,s}\Phi^T_{j,s} + (1-\bar{\phi}_j)G_{i,s|s}\breve{\bar{H}}_{i,s}\mathcal{P}^{(1)}_{ij,s|s-1}\breve{\bar{H}}^T_{j,s}\Phi^T_{j,s}\\ &+\bar{\alpha}_j\bar{\phi}_jG_{i,s|s}\breve{\bar{H}}_{i,s}\bar{\mathcal{P}}^{(1)T}_{ji,s|s|s-1}(\vec{H}_{j,s}-\vec{E}_j)^T\Phi^T_{j,s},\end{aligned}$$
$$\mathbb{E}\{G_{i,s|s}\varepsilon_{i,s}\varepsilon^T_{j,s}L^T_{j,s|s}\} = G_{i,s|s}\Xi_{ij,s}L^T_{j,s|s}.$$

Thus, (37) can be proved, which completes the proof of this theorem. ∎

A.6. Proof of Lemma 4

Proof. Substituting (14) and (45) into $\bar{\mathcal{P}}^{(1)}_{ij,s+1|s+1|s} = \mathbb{E}\{\mathcal{X}^{(1)}_{i,s+1}\tilde{\mathcal{X}}^{(1)T}_{j,s+1|s}\}$ and noting

$$\mathbb{E}\{A^{(1)}_{i,s}\mathcal{X}^{(1)}_{i,s}\varepsilon^T_{j,s}G^T_{j,s+1|s}\} = \bar{A}^{(1)}_{i,s}\mathbb{E}\{\mathcal{X}^{(1)}_{i,s}\tilde{\mathcal{X}}^{(1)T}_{j,s|s-1}\}\breve{\bar{H}}^T_{j,s}G^T_{j,s+1|s},$$
$$\mathbb{E}\{(1-\phi_{i,s})D_{i,s}\hat{\mathcal{X}}^{(1)}_{i,s|s-1}\tilde{\mathcal{X}}^{(1)T}_{j,s|s-1}A^{(1)T}_{j,s}\} = (1-\bar{\phi}_i)D_{i,s}\big(\mathbb{E}\{\mathcal{X}^{(1)}_{i,s}\tilde{\mathcal{X}}^{(1)T}_{j,s|s-1}\} - \mathcal{P}^{(1)}_{ij,s|s-1}\big)\bar{A}^{(1)T}_{j,s},$$
$$\mathbb{E}\{(1-\phi_{i,s})D_{i,s}\hat{\mathcal{X}}^{(1)}_{i,s|s-1}\varepsilon^T_{j,s}G^T_{j,s+1|s}\} = (1-\bar{\phi}_i)D_{i,s}\big(\mathbb{E}\{\mathcal{X}^{(1)}_{i,s}\tilde{\mathcal{X}}^{(1)T}_{j,s|s-1}\} - \mathcal{P}^{(1)}_{ij,s|s-1}\big)\breve{\bar{H}}^T_{j,s}G^T_{j,s+1|s},$$

it is not difficult to obtain that (38) is true. ∎

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