UNIVERSITY OF JAÉN
School of Engineering and Computing
Computer Science Department

Computing With Complex Linguistic Expressions For Decision Making Under Uncertainty

Thesis Memory Presented by

WEN HE

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to obtain the PhD Degree in Computer science

Supervisor

Dra. Rosa Mª Rodríguez Domínguez

Tutor

Dr. Luis Martínez López

Jaén, July 13, 2023

The thesis entitled *Computing with complex linguistic expressions for decision making under uncertainty*, presented by Dª Wen He to obtain the Ph.D. degree in Computer Science, has been carried out in the Computer Science Department of the University of Jaén with the supervisor Dra. Rosa Mª Rodríguez Domínguez and the tutor Dr. Luis Martínez López. To be evaluated, this research memory is presented as a set of published articles, according to Article 23, point 3, Regulation of Doctoral Studies of the University of Jaén, approved in February 2012.

Jaén, July 13, 2023

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Abstract

Decision making is a common process for human beings to carry out real-world activities, which are usually defined in situations where the information is fuzzy and imprecise. There are different approaches to dealing with this kind of uncertainty. One of the most widely used approaches in decision making is the fuzzy linguistic approach [90, 91, 92], which provides satisfactory results by modeling uncertainty using linguistic variables. The use of linguistic variables involves computing with words (CW) processes [80, 86, 94, 95]. Different computational linguistic models that model linguistic opinions can be found in the literature; however, most of them use a set of linguistic terms defined prior, which limits experts involved in decision making problems from expressing their opinions using a single linguistic term. For this reason, some researchers have pointed out that new linguistic representation models that are capable of generating more complex linguistic expressions than simple linguistic terms and that improve the flexibility of experts to express their opinions are necessary. To avoid these limitations, Rodriguez et al. [61] proposed the use of context-free grammars to generate comparative linguistic expressions (CLE) that are more flexible and richer than simple linguistic terms and closer to the cognitive model of human beings. These expressions are based on hesitant fuzzy linguistic term sets that model the experts' hesitation when they have to express their opinions and hesitate between several linguistic terms. In addition to this linguistic representation model, different computational models have been proposed. However, the loss of information in the CW processes and the lack of interpretability of the results obtained limit their use. The 2-tuple linguistic model [35] stands out in these aspects for its accuracy and interpretability of the results due to the use of symbolic translation. However, the 2-tuple linguistic values are still limited by the use of single linguistic terms. Therefore, the combination of CLE and the concept of symbolic translation could lead to improved CW processes. Recently, a new linguistic representation model for CLE has been defined along with a computational model that maintains the interpretability and accuracy of the results. This model extends the CLE through the concept of symbolic translation of the 2-tuple linguistic model, resulting in a new extended comparative linguistic expressions with symbolic translation (ELICIT) [62]. These expressions extend the representation of the CLE generated using a context-free grammar to a continuous domain to perform CW processes without any approximation. To apply this model in decision making problems, it is necessary to define new aggregation operators, similarity measures, multi-criteria decision making (MCDM) models, multi-criteria group decision making models, consensus models, etc. Therefore, in this research memory, we have presented the following proposals.

(1) Aggregation of ELICIT expressions is essential for solving decision making problems, but there are only two aggregation operators defined for aggregating ELICIT expressions. Therefore, it is necessary to define new aggregation operators in order to model a wide range of decision scenarios. The most commonly used aggregation operators in decision making are, among others, the arithmetic mean aggregation operator, the weighted mean aggregation operator, the ordered weighted mean operator, and the Choquet integral aggregation operator. These operators can be extended to aggregate ELICIT expressions, which naturally leads to the definition of new aggregation operators and new MCDM models.

(2) The ranking of ELICIT expressions is a necessary and complex task, and the challenge is to compute the similarity or distance between two ELICIT expressions. It is often calculated by the distance or similarity between their fuzzy envelopes, that is, trapezoidal fuzzy numbers. In the existing literature, different methods have been investigated to classify fuzzy numbers, for example (i) methods based on defuzzification; (ii) methods based on the distance between fuzzy numbers or (iii) pairwise comparison methods. Therefore, these methods can be applied directly to the ranking of ELICIT expressions, and then several ranking methods can be proposed.

(3) Additionally, it is necessary to take into account that there may be conflicts among experts in the group; consequently, the collective results obtained may not be accepted by all experts involved in the group decision making (GDM) problem. In other words, some experts thought that their opinions were not considered. This leads to the introduction of a consensus reaching process before linguistic decision analysis. On the other hand, it should be noted that the features of the decision problem not only define the preference structure, but also the solution scheme of the decision analysis. The most commonly used preference structures in GDM problems are preference ordering, utility vectors, reciprocal preference relations (RPR), etc. Among them, the RPR is a widely used pairwise comparison and its main advantage is that experts focus only on two alternatives at a time, which facilitates the expression of their preferences. However, this way of providing preferences limits the experts' overall perception of the alternatives, and the preferences provided could be inconsistent. Moreover, if the number of alternatives to be compared is very large, the possibility of being inconsistent increases considerably. Inconsistency in decision making leads to irrational and unreliable results. Therefore, it is important, even critical, to study the conditions under which a RPR satisfies the consistency in

GDM problems dealing with ELICIT expressions.

These proposals introduce state-of-the-art improvements to this research memory and effectively address some of the current challenges in decision making.

Keyword: ELICIT expressions, aggregation operator, multi-criteria decision making, group decision making, consensus reaching process, distance measure, similarity measure

Contents

Chapter 1

Introduction

1.1 Motivation

When solving real-world group decision making (GDM) problems, classic decision analysis [63] seeks to find the best solution from a set of alternatives based on the opinions of a group of experts. It is well known that the solution to the classical decision analysis consists essentially of two phases (see Figure 1.1) [63].

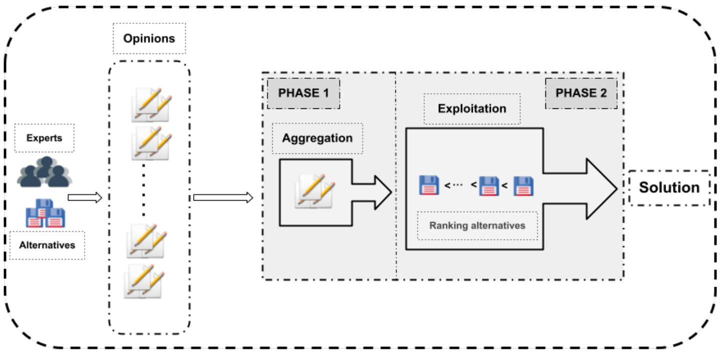

Figure 1.1: The basic scheme of solutions for classical decision analysis.

(1) **Aggregation phase:** The fusion of experts' opinions is carried out by an aggregation operator to generate a collective opinion.

(2) **Exploitation phase:** Based on the collective opinion obtained in the previous aggregation phase, a ranking, classification, or selection is then made among the alternatives and the best is selected as a solution to the problem.

However, when opinions cannot be expressed numerically in qualitative situations, it is useful to utilize linguistic variables in GDM problems. Therefore, classical decision analysis can be extended to linguistic decision analysis (LDA) [33], which can be used to solve GDM problems in cases where experts use linguistic variables to express their opinions. Hence, the solution for LDA consists of the following three steps (See Figure 1.2) [33]:

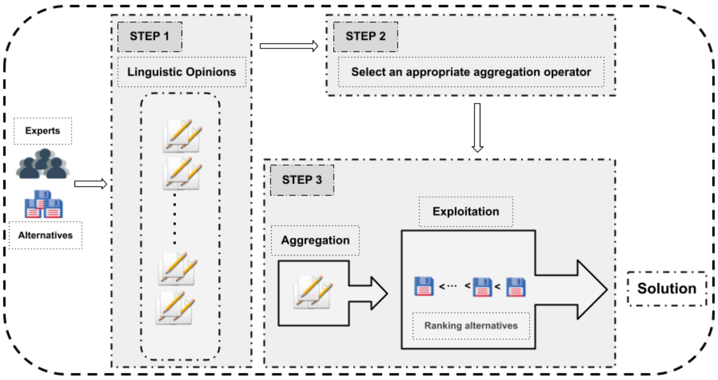

Figure 1.2: The general solution scheme of linguistic decision analysis

Step 1: To select a linguistic term set with appropriate granularity, syntax, and semantics [34], and then experts use them to express their opinions.

Step 2: An appropriate aggregation operator must be selected to fuse the provided linguistic opinions.

Step 3: In the following two phases, classical decision analysis is included but adapted to deal with linguistic opinions:

(a) **Aggregation phase:** The linguistic opinions are aggregated by a selected aggregation operator to obtain a collective linguistic opinion.

(b) **Exploitation phase:** Based on collective linguistic opinion, a ranking order is obtained among the alternatives to choose the best.

For Step 1 of LDA, once the linguistic term set has been predetermined, linguistic variables can be used to elicit the linguistic opinions of experts. Due to the inherent uncertainty of the linguistic variables, the use of fuzzy linguistic approaches (FLA) [90, 91, 92], i.e. fuzzy set theory-based approximation techniques [96], is necessary and imperative. In the existing literature, there are several FLA-based linguistic computation models to model uncertainty that follow the scheme of computing with words (CW) [80, 86, 94, 95] (see Figure 1.3), emphasizing the significance of the translation and retranslation processes in CW. For example, Herrera and Martínez [35] presented the 2-tuple linguistic model with a symbolic translation, which is intended to obtain accurate and interpretable results. However, it has limitations because it uses a single linguistic term to reflect the expert's knowledge,

Figure 1.3: CW scheme

which is sometimes insufficient. To overcome this omission, some approaches have attempted to generate more flexible and richer linguistic expressions than single linguistic terms, but most of these models provide linguistic expressions that are far from human reasoning, or do not formally define the way in which expressions are established. Thence, Rodríguez et al. [61] proposed the use of a context-free grammar to generate comparative linguistic expressions (CLE) that follow FLA and are based on the concept of hesitant fuzzy linguistic term sets (HFLTS) [60], which are capable of modeling the linguistic opinions elicited by experts when they hesitate between multiple linguistic terms. However, shortcomings of existing computational models and their extensions in terms of interpretability of results and loss of information in the CW processes may limit their use. To address these disadvantages, a novel linguistic representation model, as well as a computational model, the so-called extended comparative linguistic expressions with symbolic translation (ELICIT) [62], has been proposed. This model improves the accuracy of the results because it extends CLE to the continuous domain by means of 2-tuple linguistic values with symbolic translations, without any approximation in the CW processes, and maintains the comprehensibility and interpretability of the results. Therefore, it is beneficial to use the new ELICIT linguistic model to elicit the linguistic opinions of experts compared to other existing linguistic models. Therefore, LDA is redefined as ELICIT-DA, as shown in Figure 1.4:

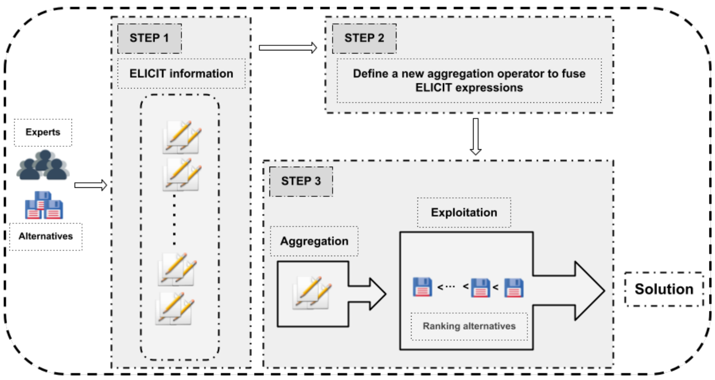

Figure 1.4: The solution scheme of ELICIT-DA

Obviously, the second step of ELICIT-DA requires the selection of aggregation operators to fuse ELICIT expressions, and then the GDM problem can be solved by performing the third step of ELICIT-DA. According to current research on aggregation operators with different features, such as arithmetic mean aggregation operator, weighted average aggregation operator, ordered weighted average (OWA) operator [83], and Choquet integral aggregation operator [14, 31] are more prominent and widely used in the solution schemes of classical decision analysis. Therefore, these aggregation operators can also be applied to fusing ELICIT expressions, which naturally lead to the definition of novel aggregation operators for ELICIT expressions.

However, the collective results obtained may not be accepted by all experts [36]. In other words, some experts thought that their opinions weren't satisfactorily considered. Therefore, a consensus reaching process (CRP) needs to be added before the third step of the ELICIT-DA. On the basis of the current research on CRP, it seems necessary to investigate the new challenges of applying CRP to deal with ELICIT expressions. Another challenge of CRP is to study the distance measure because all consensus models so far use symmetric distance measures; however, it seems interesting to analyze the impact of asymmetric distance on CRP.

1.2 Objectives

In light of the motivation and considerations presented in the previous section, the aim of this Ph.D. thesis is mainly to investigate and define new aggregation operators and decision models to solve GDM problems using ELICIT expressions. With this goal in mind, we set the following objectives:

(1) Since the ELICIT linguistic model has been proposed recently, there are a few aggregation operators that fuse ELICIT expressions. Due to the importance of using aggregation operators in the ELICIT-DA solution scheme, it is necessary to propose new fuzzy aggregation operators to fuse ELICIT expressions without loss of information and with the goal of obtaining accurate and easily understandable results.

(2) Multi-criteria decision making (MCDM) models are widely used decision models to solve GDM problems when multiple criteria are used in the decision process. Therefore, a new fuzzy MCDM model is proposed to select the best alternative/s as a solution to this problem by aggregating ELICIT expressions for each alternative evaluated according to the criteria through a predefined aggregation operator.

(3) Based on the fact that ELICIT expressions and trapezoidal fuzzy numbers can be equivalently transformed by means of fuzzy envelopes and the fact that dis-

tance/similarity measures are often used to define consensus measures or ranking orders, novel distance/similarity and asymmetric distance/similarity measures for ELICIT expressions are proposed.

(4) On the basis of the new distance/similarity measures, a novel consensus model is defined to deal with ELICIT expressions and investigate their performance. Some consensus models lead to illogical or biased results because they do not consider the consistency of the preferences provided by experts [8, 13, 38, 48, 77], therefore, a new consensus model that takes into account the consistency of the experts' preferences to handle ELICIT expressions in GDM problems is introduced.

1.3 Structure

In order to achieve the objectives set out in the previous section, this research memory will be presented in the form of a collection of articles published by doctoral students, taking into account article 23, point 3, of the current regulations for Doctoral Studies at the University of Jaén, in accordance with the program established in RD. 99/2011.

Three articles have been published in different international journals included in the JCR database produced by ISI, an article has been accepted in another international journal included in the JCR database, one article has been submitted to an international journal included in the JCR database which is under review, a book chapter has been published by CRC Press, and two international conference contributions were also published: (i) the third Intelligent and Fuzzy Systems Conference, which was published as part of the Lecture Notes in Networks and Systems book series; (ii) the IEEE International Conference on Fuzzy System 2021 classified as Core A. In summary, this report consists of a total of five contributions that cover articles that have been published, accepted, or submitted to reputable international journals, one book chapter, and two international conference contributions.

Next, we provide a brief summary of the structure of this research memory.

- **Chapter 2:** It introduces preliminary concepts used to achieve the goals in our proposal, such as decision making, decision making under uncertainty, linguistic decision making, group decision making, consensus reaching processes, multi-criteria decision making, linguistic computational models, aggregation operators, and symmetric or asymmetric distance/similarity measures.

- **Chapter 3:** It briefly presents the published/accepted articles and one under review, as well as a book chapter and two contributions from international conferences, which constitute the memory of the study. For each contribution, it is briefly discussed the results obtained.

- **Chapter 4:** In this chapter, which is the core of the Ph.D. thesis, the contributions obtained as a result of the study are included and for each of them, information on the quality index is indicated.

- **Chapter 5:** This chapter identifies and draws the final conclusions of this research, and discusses some future work as part of the development of the current study.

Chapter 2

Basic Concepts and Background

This chapter presents a brief summary of the theoretical concepts and background relevant to this thesis research. Initially, we will introduce the essential concepts of decision making, group decision making, and consensus reaching processes. Subsequently, we will revise some basic concepts about fuzzy logic, fuzzy linguistic approach, and linguistic decision making. Afterward, we will review the most important linguistic computational models that model uncertainty by means of linguistic expressions, as well as some aggregation operators necessary to fuse information and obtain collective values. Finally, we will introduce the definition of similarity measures, which are very important in CRP.

2.1 Decision making

Decision making is an activity that occurs frequently in human daily life. For example, you decide which mobile phone to buy, which movie to watch, or in which restaurant to have lunch. Due to the different natures of decision making, there are different categories into which it can be classified.

(1) Depends on the number of experts involved in the decision making process. In case only one person is involved in the decision problem, it is referred to as individual decision making. However, in most decision making problems, several experts are usually involved, which is referred to as group decision making or multi-expert decision making.

(2) Depends on the number of criteria. There are decision making problems that imply a simple optimization of the alternatives according to a single criterion, while others require multiple optimizations based on multiple criteria, which are referred to as multi-criteria decision making.

In short, this research memory will focus on linguistic decision problems with multiple criteria and/or multiple experts.

2.1.1 Group decision making

GDM is a typical decision making situation involving two or more experts who provide their preferences and reach a collective decision with their knowledge and attitude toward the decision problem [51]. In a GDM problem, a group of experts $\mathcal{E} = \{e_1, e_2, \ldots, e_K\}$ $(K \geq 2)$, using preferences structures such as preference ordering [72], utility vectors [37], reciprocal preference relations (RPRs) [42, 64], etc., are asked to express their opinions on a set of possible alternatives $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ $(n \geq 2)$. Among these, RPR is widely used in such GDM problems. For all $\kappa \in \{1, 2, \ldots, K\}$, a RPR is often constructed and denoted as follows:

$$P_\kappa = \begin{pmatrix} p_{11,\kappa} & \cdots & p_{1i,\kappa} & \cdots & p_{1j,\kappa} & \cdots & p_{1n,\kappa} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ p_{i1,\kappa} & \cdots & p_{ii,\kappa} & \cdots & p_{ij,\kappa} & \cdots & p_{in,\kappa} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ p_{j1,\kappa} & \cdots & p_{ji,\kappa} & \cdots & p_{jj,\kappa} & \cdots & p_{jn,\kappa} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ p_{n1,\kappa} & \cdots & p_{ni,\kappa} & \cdots & p_{nj,\kappa} & \cdots & p_{nn,\kappa} \end{pmatrix}$$

being $p_{ij,\kappa}$ the preference provided by a expert e_κ for the alternatives' pair $\langle \mathcal{A}_i, \mathcal{A}_j \rangle$, interpreted as the preference degree of the alternative $\mathcal{A}_i$ over $\mathcal{A}_j$.

Due to the different relationships between $p_{ij,\kappa}$ and $p_{ji,\kappa}$ of P_κ for all $i, j \in \{1, 2, \ldots, n\}$, there are two types of RPR:

(1) An additive RPR [38] that satisfies $p_{ij,\kappa}, p_{ji,\kappa} \in [0, 1]$ and $p_{ij,\kappa} + p_{ji,\kappa} = 1$. In this way, the following interpretation is usually assumed:

  (a) $p_{ij} = 0$ indicates $p_{ji} = 1$, i.e., $\mathcal{A}_j$ is absolutely preferred to $\mathcal{A}_i$;

  (b) $p_{ij} \in (0, 0.5)$ indicates $\mathcal{A}_i \prec \mathcal{A}_j$, i.e., $\mathcal{A}_j$ is preferred to $\mathcal{A}_i$;

  (c) $p_{ij} = 0.5$ indicates $\mathcal{A}_i \sim \mathcal{A}_j$, i.e., $\mathcal{A}_i$ is indifference preferred to $\mathcal{A}_j$;

  (d) $p_{ij} \in (0.5, 1)$ indicates $\mathcal{A}_i \succ \mathcal{A}_j$, i.e., $\mathcal{A}_i$ is preferred to $\mathcal{A}_j$;

  (e) $p_{ij} = 1$ indicates $\mathcal{A}_i$ is absolutely preferred to $\mathcal{A}_j$.

(2) A multiplicative RPR [64] that satisfies $p_{ij,\kappa}, p_{ji,\kappa} \in \left[\frac{1}{9}, 9\right]$ and $p_{ij,\kappa} \times p_{ji,\kappa} = 1$. Similarly, the following interpretation is usually assumed:

  (a) $p_{ij} = \frac{1}{9}$ indicates $\mathcal{A}_j$ is absolutely preferred to $\mathcal{A}_i$;

(b) $p_{ij} \in \left(\frac{1}{9}, 1\right)$ indicates $\mathcal{A}_i \prec \mathcal{A}_j$, i.e., $\mathcal{A}_j$ is preferred to $\mathcal{A}_i$;

(c) $p_{ij} = 1$ indicates $\mathcal{A}_i \sim \mathcal{A}_j$, i.e., $\mathcal{A}_i$ is indifference preferred to $\mathcal{A}_j$;

(d) $p_{ij} \in (1, 9)$ indicates $\mathcal{A}_i \succ \mathcal{A}_j$, i.e., $\mathcal{A}_i$ is preferred to $\mathcal{A}_j$;

(e) $p_{ij} = 9$ indicates $\mathcal{A}_i$ is absolutely preferred to $\mathcal{A}_j$.

A RPR is a widely used pairwise comparison, and its main advantage is that experts focus only on one pair of alternatives at a time, which facilitates the expression of their preferences. However, this way used to provide preferences limits the overall perception of experts of the alternatives, and such preferences might be irrational. In addition, irrationality is usually associated with a low consistency of RPR, which is generally represented by the transitivity property [15, 16, 38, 70]. Therefore, both low consistency of RPR and lack of consistency in decision making can lead to irrational and misleading conclusions, which leads to the conclusion that it is important, necessary, and even critical to study the conditions under which RPR satisfies consistency [12, 38, 64]. There are two acceptable transitivity properties that characterize consistency, namely the additive transitivity property and the multiplicative transitivity property [38]. The multiplicative transitivity property of the multiplicative RPR can be equivalently transformed into the additive transitivity property of the additive RPR [12, 38]. Therefore, in this research memory, we focus on additive RPRs that satisfy the additive transitivity property to avoid misleading results.

To better understand the relationship between the additive transitivity property and the consistency of additive RPRs, the following definition of additive consistency of RPRs is provided.

Definition 1 *[12, 38, 71, 72] Let an additive RPR, $P = (p_{ij})_{n\times n}$, be a matrix defined over a set of alternatives $\mathcal{A} = \{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$, $n \geq 2$, then this matrix is consistent if and only if the additive transitivity property is satisfied:*

$$p_{ij} + p_{jk} - p_{ik} = \frac{1}{2},\ \forall i, j, k \in \{1, 2, \ldots, n\} \tag{2.1}$$

Therefore, it seems necessary to study the consistency of RPRs before applying the solution scheme of a GDM problem using RPR preferences. It consists of following two steps:

Step 1: Consistency Check: It compares the current consistency index with a predefined consistency threshold.

(a) If the current consistency index is less than the threshold, it is referred to as an unacceptable RPR, and then its consistency index needs to be improved until the threshold is reached;

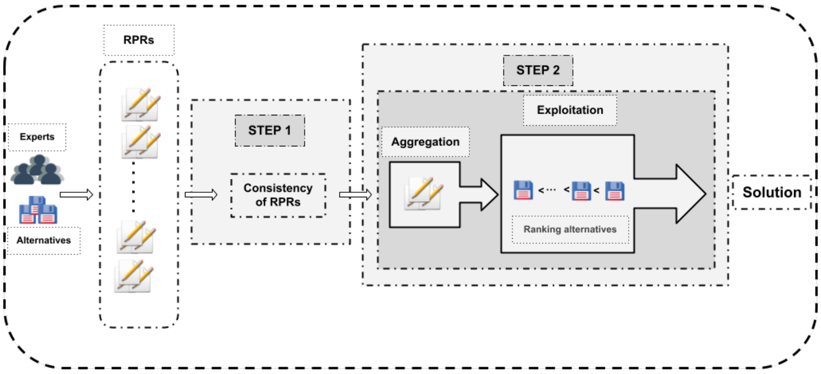

Figure 2.1: The general solution scheme for a GDM problem using RPR preferences

(b) otherwise, go to the decision analysis directly.

Step 2: Subsequently, decision analysis is incorporated, but extended to address the GDM problem of RPR preferences:

(a) **Aggregation phase:** A GDM problem dealing with RPRs is composed of two steps:

(a-1) The RPRs, $P_\kappa = (p_{ij,\kappa})_{n\times n}$, provided by experts are fused by an aggregation operator to obtain a collective RPR, $P_c = (p_{ij,c})_{n\times n}$, as follows:

$$p_{ij,c} = \sum_{\kappa=1}^{K} w_\kappa \cdot p_{ij,\kappa} \tag{2.2}$$

being $w_\kappa \geq 0$ the weight of the κ^{th} expert and $\sum_{\kappa=1}^{K} w_\kappa = 1$.

(a-2) And then for each alternative, the collective result is computed by another aggregation operator (it could be the same as in the previous step) as follows:

$$p_{i,c} = \sum_{j=1}^{n} \omega_j \cdot p_{ij,c} \tag{2.3}$$

being ω_j the weight of alternative $\mathcal{A}_j$ such that $\omega_j \geq 0$ for all $j \in \{1, 2, \ldots, m\}$ and $\sum_{\kappa=1}^{m} \omega_j = 1$.

(b) **Exploitation phase:** Based on collective opinions $p_{i,c}$ for all $i \in \{1, 2, \ldots, n\}$, a ranking, classification, or selection is obtained among the alternatives; and then the best alternative, $\max_i \{p_{i,c}\}$, is selected as the solution to the GDM problem.

2.1.2 Consensus reaching process

Traditionally, the GDM problem solving process consists only of a selection process [63] aimed at finding a collective solution based on the preferences or opinions of a group of experts. However, due to potential conflicts among experts within the group, the final result may not be accepted by all participating expert. To achieve a high level of agreement and eliminate conflicts, it is necessary to include the CRP prior to the selection process in order to obtain a mutually agreeable collective solution. In other words, decisions that may affect a group of people are better accepted when all experts involved in the GDM problem agree [36].

The concept of consensus has been interpreted differently by different experts, ranging from complete or unanimous agreement (which is difficult to achieve in practice) to a more flexible interpretation. For example, Saint and Lawson [65] expressed consensus as a "*state of mutual agreement between all members of the group, in which all legitimate concerns of each individual have been taken into account for collective satisfaction*". This is a unanimous consensus that cannot exist in the real-world. Then, kacprzyk and Fedrizzi [43, 44] introduced an easier-to-implement concept of "*soft*" consensus, which is based on the concept of a fuzzy majority, according to which consensus exists when "*most of the important experts agree on most of the relevant options*" [41, 42]. This is more human-consistent and appropriate to reflect the human view of the meaning of consensus. The "*soft*" consensus-based CRP is an evolving iterative negotiation process consisting of several rounds of discussions to reach a mutual agreement among the majority of experts [26]. Thus, to achieve such a "soft" consensus, the GDM problem-solving process consists of two main processes: (i) a CRP and (ii) a selection process. The classical CRP that deals with RPR for GDM can be divided into 5 phases, as described below and shown in Figure 2.2.

Phase 1: **Framework configuration:** For a defined GDM problem, the predefined consensus threshold δ and maximum rounds t_{max} are initially described.

Phase 2: **Gathering RPRs:** Each expert provides his opinions using an additive RPR matrix.

Phase 3: **Computing the current consensus degree:** The current consensus level, CD^t, is often calculated by distance and aggregation operators [4, 58].

Phase 4: **Consensus control:** The current consensus degree CD^t is compared to a predefined threshold δ. If $CD^t \geq \delta$ or $t \geq t_{max}$, a selection process

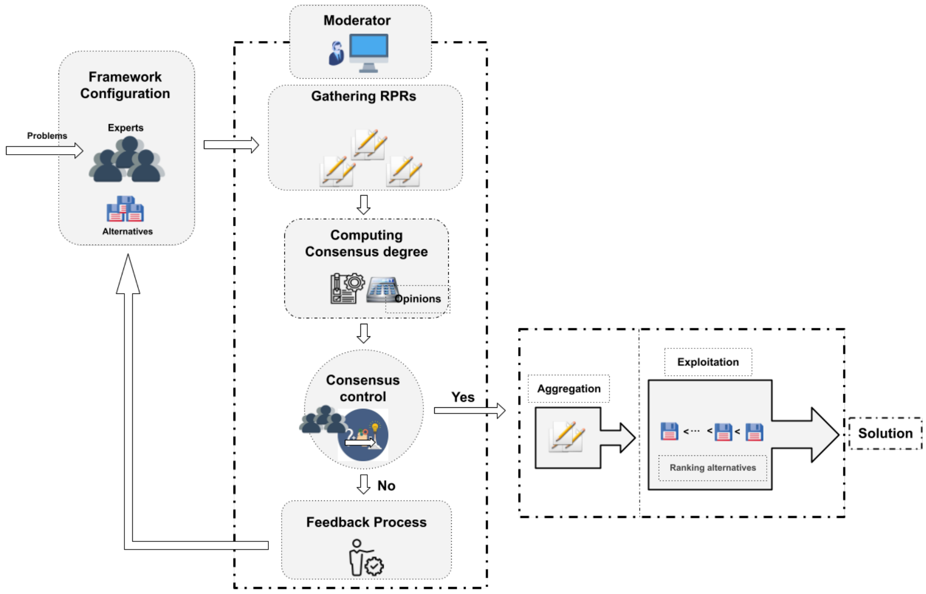

Figure 2.2: The general solution scheme of CRP for GDM dealing with RPRs

is applied to choose the best alternative/s; otherwise, a new round of discussion and negotiation is required.

Phase 5: **Feedback process:** Before proceeding with a new round of CRP, it is important to identify the opinions that caused the disagreement and then generate some recommendations to guide experts in revising the opinions to bring them closer to increasing the consensus level in the next round.

As we have mentioned above, both the lack of consistency and the low consistency of RPR in decision making can lead to irrational and misleading conclusions [12, 38]. Therefore, the consistency of RPRs should be considered before CRP to solve a GDM problem.

2.1.3 Multi-criteria decision making

The classical multi-criteria decision making (MCDM) problem involves multiple criteria in which multiple experts usually express their opinions on a set of finite alternatives according to multiple criteria in order to rank these alternatives and select the best one as the solution to the problem. Formally, the MCDM problem is often defined as follows:

(1) A group of multiple experts $\mathcal{E} = \{e_\kappa \mid \kappa \in \{1, \ldots, K\}\}$, $K \geq 2$;

(2) A set of finite alternatives $\mathcal{A} = \{\mathcal{A}_i \mid i \in \{1, \ldots, n\}\}$, $n \geq 2$;

(3) A set of multiple criteria $\mathcal{C} = \{\mathcal{C}_j \mid j \in \{1, \ldots, m\}\}$, $m \geq 2$;

The expert evaluates the alternatives against the criteria by the use of a decision matrix, for all $\kappa \in \{1, 2, \ldots, K\}$, it is expressed as follows:

$$M_\kappa = \begin{pmatrix} m_{11,\kappa} & \ldots & m_{1j,\kappa} & \ldots & m_{1m,\kappa} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ m_{i1,\kappa} & \ldots & m_{ij,\kappa} & \ldots & m_{im,\kappa} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ m_{n1,\kappa} & \ldots & m_{nj,\kappa} & \ldots & m_{nm,\kappa} \end{pmatrix}$$

being $m_{ij,\kappa}$ the preference provided by the expert e_κ over the alternatives $\mathcal{A}_i$ according to criteria $\mathcal{C}_j$.

The general MCDM model for solving such problems follows the general scheme of the GDM problem and also contains two main phases (See Figure 2.3):

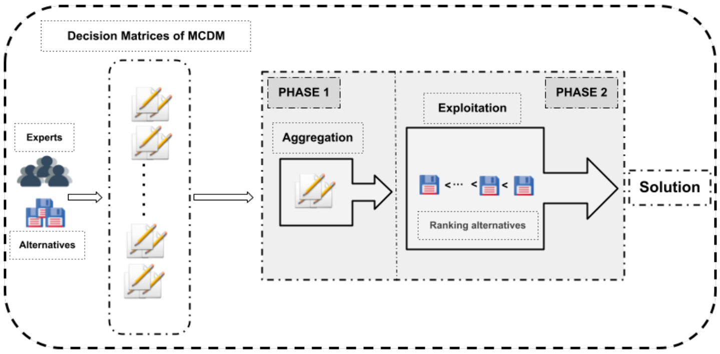

Figure 2.3: The general solution scheme of classical decision analysis

(1) **Aggregation phase:** In this phase, a MCDM model for a GDM problem consists of two steps:

Step 1: For each position (i, j) of the decision matrices $M_\kappa = (m_{ij,\kappa})_{n \times n}$ for all $\kappa \in \{1, \ldots, K\}$, these values are aggregated to obtain a collective value of the collective decision matrix $M_c = (m_{ij,c})_{n \times n}$. The fusion of $m_{ij,\kappa}$ is defined as follows:

$$m_{ij,c} = \sum_{\kappa=1}^{K} w_\kappa \cdot m_{ij,\kappa} \qquad (2.4)$$

being $w_\kappa \geq 0$ the weight of κ^{th} expert and $\sum_{\kappa=1}^{K} w_\kappa = 1$.

Step 2: For each alternative, that is, for each row of the decision matrix $M_c = (m_{ij,c})_{n \times n}$, the values of $m_{ij,c}$ for all $j \in \{1, \ldots, n\}$ are aggregated by another aggregation operator to obtain the collective value $m_{i,c}$, as follows:

$$m_{i,c} = \sum_{j=1}^{n} \omega_{\mathcal{C}_j} \cdot m_{ij,c} \tag{2.5}$$

being $\omega_{\mathcal{C}_j} \geq 0$ the weight of criteria $\mathcal{C}_j$ and $\sum\limits_{j=1}^{m} \omega_{\mathcal{C}_j} = 1$.

(2) **Exploitation phase:** Based on the collective opinion obtained in the previous aggregation phase, a ranking, classification, or selection is then obtained among the alternatives, $m_{i,c}$ for all $i \in \{1, 2, \ldots, n\}$; and the best one, $\max\limits_{i} \{m_{i,c}\}$, is then selected as the solution to the problem.

2.2 Decision making under uncertainty

This section introduces some basic concepts related to uncertainty decision making, such as fuzzy logic and the fuzzy linguistic approach [90, 91, 92], which use linguistic variables to model uncertainty and provide successful results. Furthermore, it also reviews several linguistic computational models that are necessary to understand our proposal.

2.2.1 Fuzzy logic theory and fuzzy linguistic approach

Fuzzy logic theory [88] was originally proposed by Zadeh with the aim of modeling uncertainty or imprecision. To do so, he extended the definition of crisp sets to fuzzy sets, considering the case where the boundaries of their sets are not strictly defined.

To better explain fuzzy logic theory, we first review the definition of a crisp set, which is defined as follows:

Definition 2 *[45] Let X be the discourse universe. A crisp set $A \subset X$ is often denoted as $A = \{x \,|x \in A\}$. Then the characteristic function associated with A shall be defined as a mapping* $\begin{array}{l} A(x) \ : X \longrightarrow \{0,1\} \\ x \mapsto A(x) \end{array}$ *such that*

$$A(x) = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{if } x \notin A. \end{cases} \tag{2.6}$$

The definition of a fuzzy set is slightly different from that of a crisp set, but its interpretation is quite different. Fuzzy sets are often used to model uncertainty and are defined by extending the function values from $\{0,1\}$ to $[0,1]$ by replacing the membership functions with the characteristic functions of the crisp set. It is defined as follows:

Definition 3 *[89] Let X be the discourse universe. A fuzzy set $\tilde{A}$ is defined in X, whose membership function $\tilde{A}(x)$ is defined as a mapping* $\begin{array}{c} \tilde{A}(x) \ : X \longrightarrow [0,1] \\ x \mapsto \tilde{A}(x) \end{array}$ *such that*

$$\tilde{A} = \left\{ \left(x, \tilde{A}(x) \right) \middle| \ x \in X, \tilde{A}(x) \in [0,1] \right\} \tag{2.7}$$

It is worth noting that the fuzzy set has several interesting points.

(1) It is evident that the value of $\tilde{A}(x)$ can characterize more complex cases than $A(x)$ because it has membership values between 0 and 1, not only 0 or 1.

(2) The closer the value is to 1, the higher the value of $\tilde{A}(x)$, that is, the higher the degree of membership of x belonging to $\tilde{A}$. On the other hand, the closer it is to 0, the lower the degree of membership of x belonging to $\tilde{A}$.

Therefore, it is necessary to define an appropriate membership function to characterize the degree of membership of x belonging to the fuzzy set.

Furthermore, various types of functions can be used to characterize the membership functions of a fuzzy set, such as *L-R* membership functions [67], that is, quasi-trapezoidal fuzzy numbers [19]. Among existing functions, a trapezoidal fuzzy number (TrFN) [89] (triangle fuzzy number (TFN) [89] is a special case of TrFN) is a widely used membership function, whose definition is given as follows.

Definition 4 *[89] Let* $\mathbf{R}$ *be a set of all real numbers. A TrFN is often denoted as* $T \equiv T(a,b,c,d)$ *with* $a \le b \le c \le d$ *for all* $a,b,c,d \in \mathbf{R}$ *and is defined as a mapping* $\begin{array}{c} T \ : \mathbf{R} \longrightarrow [0,1] \\ x \mapsto T(x) \end{array}$ *as follows:*

$$T(x) = \begin{cases} \frac{x-a}{b-a} & \text{if } a < x < b \\ 1 & \text{if } b \le x \le c \\ \frac{d-x}{d-c} & \text{if } c < x < d \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

Furthermore, if $b = c$, then T is a TFN.

Since most real-world decision problems are usually defined under uncertainty, it is necessary to propose decision models based on fuzzy logic [93, 96], which plays a fundamental role in the successful solution of these problems. In such uncertainty cases, FLA [90, 91, 92] has been successfully used in linguistic modeling [52] by using "*linguistic variables*" to represent linguistic information. "*linguistic variables*" [90, 91, 92] were an essential concept initially presented by Zadeh, which are "*variables whose values are not numbers, but words or sentences in natural or artificial language*". Although it does not compute as precisely as numbers, it is successfully used to express human opinions and more closely resembles human cognitive processes. It is defined as follows:

Definition 5 *[90, 91, 92] A linguistic variable is characterized by a quintuple* $(H, T(H), U, G, M)$*, where* H *denotes the name of the variable;* $T(H)$ *denotes the set of terms of* H*, i.e. the set of its linguistic values;* U *denotes a universe of discourse;* G *is the syntactic rule that generates the terms in* $T(H)$*;* M *is a semantic rule whose meaning is associated with each linguistic value, and* $M(X)$ *denotes a fuzzy subset of* U*.*

For ease of notation, let $S = \left\{s_0, s_1, \ldots, s_{\frac{g}{2}}, \ldots, s_g\right\}$ be a linguistic term set with its semantics having odd $(g+1)$ granularity. For instance, a linguistic term set with its semantics is shown in Figure 2.4.

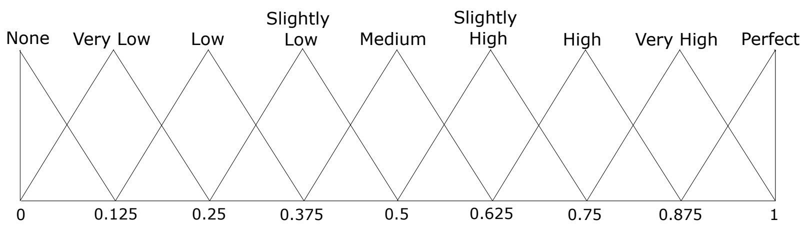

Figure 2.4: A 9-terms set with its semantics

Then, the $g+1$ terms set with its semantics satisfies the following properties:

(P1) It has an inherent naturally order: $s_i \leq s_j$ if and only if $i \leq j$;

(P2) It has a negative operator $\boldsymbol{Neg}(\cdot)$ and is given as: $s_i = \boldsymbol{Neg}(s_{g-i})$.

2.2.2 Linguistic decision analysis

Decision problems are usually defined in the setting of fuzzy or imprecise information. In these cases, experts express their preferences through linguistic information [90, 91, 92] rather than using precise numbers. Thence, classical decision analysis can be extended to LDA [33], which can be used to solve linguistic GDM problems, which consists of three steps (See Figure 1.2) [33]:

Step 1: To select an appropriate linguistic term set with appropriate granularity, syntax, and semantics [34], and then experts use it to express their opinions.

Step 2: The focus of this step is to select a suitable aggregation operator that fuse linguistic opinions in the next selection process.

Step 3: Following the classical decision analysis, a selection process is performed to obtain a final solution to this problem.

The solution scheme shown in Figure 1.2 indicates that to find a solution to the linguistic GDM problem using LDA, it is necessary to manipulate the linguistic opinions. It is in this sense that the CW processes [54, 80, 86, 93] are needed to simulate the human reasoning process to produce linguistic results from linguistic premises.

Throughout the research memory, the focus will be on the computational process performed by the CW that deals with linguistic decision problems. Due to the wide and intensive use of linguistic information [22, 52], different CW schemes [32, 54, 53] have been developed. However, all of these highlight the need to obtain accurate and understandable linguistic results. To achieve such results, Yager [80, 86] presented a CW scheme that consists of three main processes, namely translation, manipulation, and retranslation (see Figure 1.3).

(1) **Translation process**: The process of translating input linguistic opinions into fuzzy logic-based formats;

(2) **Manipulation process**: The computations in this process are usually performed using fuzzy tools, and the results obtained are then applied to the subsequent retranslation process;

(3) **Retranslation process**: In contrast to the translation process, it is a process of retranslating the collective outcomes aggregated in the previous process into a linguistic format that can be easily interpreted and understood by humans.

2.3 Linguistic computational models

The previous section has pointed out that modeling uncertainty through linguistic opinions implies a CW process, as described in the previous section. Therefore, numerous computational linguistic models have been presented in the literature. In this section, we recall the models that are most relevant to the research in the memory of this Ph.D. thesis.

2.3.1 2-tuple linguistic model

This subsection briefly reviews the concepts related to the 2-tuple linguistic model proposed by Herrera and Martínez [35], which addresses the shortcomings in precision and interpretability of the classical model based on computational methods, such as the extension principle [6, 17] and the symbolic method [18], referring to the linguistic domain as continuous and performing calculations without any approximation.

For the sake of notation, let $\overline{S} \equiv S \times [-0.5, 0.5)$ be a set of all 2-tuple linguistic values, then $(s_i, \alpha) \in \overline{S}$ is used to express expert's opinion, where $s_i \in S$ denotes a linguistic term and $\alpha \in [-0.5, 0.5)$, also known as symbolic translation, indicating its displacement value with respect to s_i. Considering that S is a finite set of linguistic terms, the value of α has some special cases, as follows:

$$\alpha \in \begin{cases} [0, 0.5) & \textit{if } s_i = s_0 \\ [-0.5, 0.5) & \textit{if } s_i \in \left\{ s_1, \ldots, s_{\frac{g}{2}}, \ldots, s_{g-1} \right\} \\ [-0.5, 0] & \textit{if } s_i = s_g \end{cases} \tag{2.9}$$

For example, in Fig. (2.5), $x_1 = (s_3, 0)$, $x_2 = (s_3, -0.5)$, $x_3 = (s_2, 0.25)$ and $x_4 = (s_3, -0.25)$ are 2-tuple linguistic values defined on a set of 7 terms with its semantics.

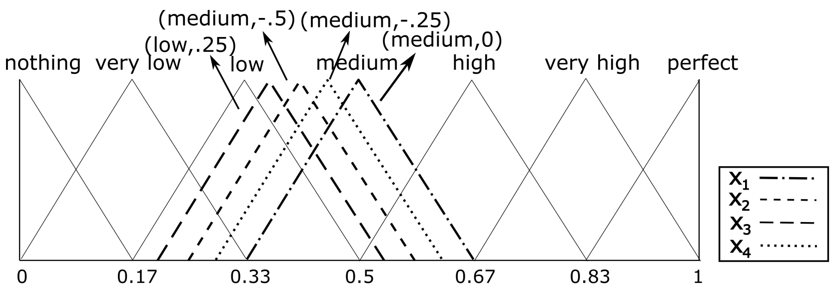

Figure 2.5: Some 2-tuple linguistic values defined on a 7 terms set with its semantics

As described in the literature [35, 52], the 2-tuple linguistic values can usually be processed as real numbers and vice versa by the following functions.

Definition 6 *[35, 52] The function* $\begin{array}{c}\Delta_S^{-1} : \overline{S} \longrightarrow [0, g] \\ (s_i, \alpha) \mapsto \beta\end{array}$ *given by*

$$\Delta_S^{-1}(s_i, \alpha) = \beta = i + \alpha \tag{2.10}$$

is a bijection with regard to the set S*, and its inverse function* $\begin{array}{r}\Delta_S : [0, g] \longrightarrow \overline{S} \\ \beta \mapsto (s_i, \alpha)\end{array}$

is then defined as

$$\Delta_S(\beta) = (s_{round(\beta)}, \beta - round(\beta)),\ \forall \beta \in [0, g], \tag{2.11}$$

being assigned by $round(\cdot)$ *to the nearest integer* $i \in \{0, 1, \ldots, g\}$.

Proposition 1 *Once the functions* $\Delta_S(\cdot)$ *and* $\Delta_S^{-1}(\cdot)$ *for 2-tuple linguistic computation model are given, the 2-tuple linguistic value satisfies the following properties [35, 52]:*

(P1) *Let* (s_i, α_1) *and* (s_j, α_2) *be two 2-tuple linguistic values for all* $i, j \in \{0, 1, \ldots, g\}$ *and* $\alpha_1, \alpha_2 \in [-0.5, 0.5)$, *then they are ordered by using the following rules:*

- *(a)* *If* $s_i > s_j$, *then* $(s_i, \alpha_1) \succ (s_j, \alpha_2)$;
- *(b)* *If* $s_i = s_j$,
  - *i.* *if* $\alpha_1 > \alpha_2$, *then* $(s_i, \alpha_1) \succ (s_j, \alpha_2)$;
  - *ii.* *if* $\alpha_1 = \alpha_2$, *then* $(s_i, \alpha_1) = (s_j, \alpha_2)$;
  - *iii.* *if* $\alpha_1 < \alpha_2$, *then* $(s_i, \alpha_1) \prec (s_j, \alpha_2)$.

(P2) *And the negative operator* $\boldsymbol{Neg}(\cdot)$ *is given as:* $\boldsymbol{Neg}(s_i, \alpha_1) = \Delta_S\left(g - \Delta_S^{-1}(s_i, \alpha_1)\right)$.

2.3.2 Hesitant fuzzy linguistic term sets

In real-world linguistic DM problems, experts sometimes hesitate among multiple linguistic terms at the same time due to the challenges of eliciting their assessments or preferences for alternatives with a single linguistic term when time is limited, confidence is low, information is lacking, or other uncertainties are being considered.

To model this hesitation, Rodríguez et al. presented the concept of hesitant fuzzy linguistic term set (HFLTS) [60], which facilitates the expression of linguistic opinions.

Definition 7 *[60] Let* $S = \left\{s_0, s_1, \ldots, s_{\frac{g}{2}}, \ldots, s_g\right\}$ *be an odd* $g+1$ *granularity linguistic term set with its semantics, then a HFLTS is denoted* H_S *as it is defined as a consecutive finite subset of* S, *i.e., for all* $0 \le i \le j \le g$, $H_S \subset S$ *is given as*

$$H_S = \{s_i, s_{i+1}, \ldots, s_j\}. \tag{2.12}$$

Thence, for all $0 \le i \le j \le g$, all possible HFLTSs can be expressed as follows:

$$H_S = \begin{cases} \{s_i\} & \textit{if} \quad i = j; \\ \{s_0, s_1, \ldots, s_j\} & \textit{if} \quad i = 0; \\ \{s_i, s_{i+1}, \ldots, s_g\} & \textit{if} \quad j = g; \\ \{s_i, s_{i+1}, \ldots, s_j\} & \textit{if} \quad 1 \le i \le j \le g-1. \end{cases} \tag{2.13}$$

Remark 1 *(1) For a given S, since S itself is a universe set, it is necessary to point out two special HFLTSs that are different from the current one, namely the empty HFLTS and the full HFLTS, which are defined as follows:*

(a) the empty HFLTS: $H_S = \emptyset$;

(b) the full HFLTS: $H_S = \{s_0, s_1, \ldots, s_g\} = S$.

(2) Let Ω_{H_s} be the set including $\emptyset$, S, $S - H_S$ and all possible H_S. According to the set theory, it is natural to have some calculations and operations as follows:

(a) For a given $H_S \in \Omega_{H_S}$, the upper bound H_S^+ and the lower bound H_S^- are defined respectively as:

$$\begin{aligned} H_S^+ &= \max\{s_j |\ s_i \in H_S \ and \ s_i \leq s_j\} \\ H_S^- &= \min\{s_j |\ s_i \in H_S \ and \ s_i \geq s_j\} \end{aligned} \tag{2.14}$$

(b) For $\emptyset$ and S, it is obviously that they satisfy $\emptyset \cup S = S$ and $\emptyset \cap S = \emptyset$. This is also true for the pair $S - H_S$ and H_S, with $S - H_S$ being called the complement of H_S. And $S - H_S$ is computed as follows:

$$H_S^c = S - H_S = \{s_i |\ s_i \in S \ and \ s_i \notin H_S\} \tag{2.15}$$

Thence, $(H_S^c)^c = H_S$.

(c) $\forall H_S^1, H_S^2 \in \Omega_{H_S}$, their union and intersection are given respectively as:

$$\begin{aligned} H_S^1 \cup H_S^2 &= \left\{s_i |\ s_i \in H_S^1 \ or \ s_i \in H_S^2\right\} \\ H_S^1 \cap H_S^2 &= \left\{s_i |\ s_i \in H_S^1 \ and \ s_i \in H_S^2\right\} \end{aligned} \tag{2.16}$$

In linguistic decision problems, comparisons between several linguistic terms are needed to perform the selection process, which is defined differently due to different FLA methods. However, the comparisons between HFLTS are not straightforward because they are linguistic values, not precise numbers. Therefore, to simplify and facilitate these operations, the concept of envelope [60] was introduced for HFLTS as follows.

Definition 8 *[60] A mapping $env(\cdot) : H_S \mapsto \left[H_S^-, H_S^+\right]$ is defined as an envelope of the HFLTS H_S such that*

$$env(H_S) = \left[H_S^-, H_S^+\right], \ H_S^- \leq H_S^+ \tag{2.17}$$

where $H_S^+ = \max\{s_j |\ s_i \in H_S \ and \ s_i \leq s_j\}$ and $H_S^- = \min\{s_j |\ s_i \in H_S \ and \ s_i \geq s_j\}$.

While HFLTS can be used to model the hesitation of experts in real-world problems, they do not resemble the way humans think and reason in real-world problems. Thus, in [61], simple but refined linguistic expressions, called comparative linguistic expressions (CLE), are closer to the usage of human language. It is based on HFLTS and context-free grammar and is defined as follows:

Definition 9 *[61] Let G_H be a context-free grammar and $S = \{s_0, s_1 \dots s_g\}$ a linguistic term set, and the elements of $G_H = (V_N, V_T, I, P)$ given by:*

$$\begin{aligned}
&V_N = \{(\textit{primary term}),\ (\textit{composite term}),\ (\textit{unary relation}), (\textit{binary relation}),\\
&\qquad (\textit{conjunction})\}\\
&V_T = \{\textit{at least},\ \textit{at most},\ \textit{between},\ \textit{and},\ s_0, \dots, s_g\}\\
&I \in V_N
\end{aligned}$$

The production rules P defined in an extended Backus-Naur Form [7] are as follows:

$$\begin{aligned}
P = \{&I\ ::=\ (\textit{primary term})|(\textit{composite term})\\
&(\textit{composite term})\ ::=\ (\textit{unary relation})(\textit{primary term})|\\
&\qquad (\textit{binary relation})(\textit{primary term})(\textit{conjunction})(\textit{primary term})\\
&(\textit{primary term})\ ::=\ s_0|s_1|\dots|s_g\\
&(\textit{unary relation})\ ::=\ \textit{at least}|\textit{at most}\\
&(\textit{binary relation})\ ::=\ \textit{between}\\
&(\textit{conjunction})\ ::=\ \textit{and}\}
\end{aligned}$$

According to Definition 9, for all $0 \le i \le j \le g$, all possible CLEs [61] defined over S are denoted as ll_S and presented as:

$$ll_S = \begin{cases} s_i & \textit{if}\ \ i = 0, 1, \dots, g; \\ \textit{at most } s_j & \textit{if}\ \ 1 \le j \le g; \\ \textit{at least } s_i & \textit{if}\ \ 0 \le i \le g-1; \\ \textit{between } s_i \textit{ and } s_j & \textit{if}\ \ 1 \le i \le j \le g-1. \end{cases} \tag{2.18}$$

Therefore, CLEs can be semantically equivalent represented as HFLTSs by the transformation function E_{G_H}, and the operations on CLEs can be performed by those on HFLTS. The transformation function E_{G_H} is defined as follows:

Definition 10 *[60] Let ll_S be a CLE, generated by a context-free grammar G_H over a given linguistic term set S. A transformation function E_{G_H} is defined as a mapping*

$$E_{G_H} : ll_S \mapsto H_S \tag{2.19}$$

such that for all $0 \le i \le j \le g$, E_{G_H} *is given as follows:*

$$E_{G_H}(ll_S) = \begin{cases} \{s_i\} & \text{if } ll_S = s_i; \\ \{s_0, s_1, \ldots, s_j\} & \text{if } ll_S = \text{at most } s_j; \\ \{s_i, s_{i+1}, \ldots, s_g\} & \text{if } ll_S = \text{at least } s_i; \\ \{s_i, s_{i+1}, \ldots, s_j\} & \text{if } ll_S = \text{between } s_i \text{ and } s_j. \end{cases} \tag{2.20}$$

Similarly, because $E_{G_H}(ll_S)$ is a HFLTS, the envelope of CLE [61] is defined as a mapping $env(\cdot) : E_{G_H}(ll_S) \mapsto [H_S^-, H_S^+]$ such that

$$env(E_{G_H}(ll_S)) = [H_S^-, H_S^+], \ H_S^- \le H_S^+ \tag{2.21}$$

Based on such linguistic intervals, several different operators and models [60, 61] have been proposed to deal with HFLTS / CLE and obtain crisp values by using the function Δ_S^{-1} [33], which is no longer a fuzzy number or a linguistic value that indicates some loss of information. To avoid this, Liu and Rodríguez [50] introduced a new definition of the fuzzy envelope in the form of TrFN for HFLTS/CLE.

To save space, here we only review the general procedure for computing the fuzzy envelope of HFLTS (see [50] for further details). To compute the fuzzy envelope of $H_S = \{s_i, s_{i+1}, \ldots, s_j\}$, denoted as $env_F(H_S) \equiv T(a, b, c, d)$, a 4-step procedure is required (see Figure 2.6).

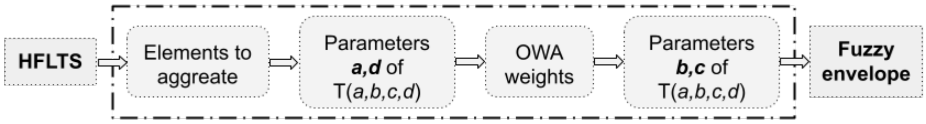

Figure 2.6: The general procedure for constructing the fuzzy envelope of HFLTS

Step 1: To identify the elements to be aggregated:

(a) For $s_k \in H_S$, it can be denoted as a TFN as $T_{s_k} = T(a_k, b_k, d_k)$;

(b) All points of $s_k \in H_S$ are denoted as the set of elements need to be aggregated as follows:

$$\left\{ \underbrace{a_i, b_i, d_i}_{s_i}, \underbrace{a_{i+1}, b_{i+1}, d_{i+1}}_{s_{i+1}}, \ldots, \underbrace{a_j, b_j, d_j}_{s_j} \right\} \tag{2.22}$$

(c) According to the definition of HFLTS, it obtains $b_k = a_{k+1}$ and $d_k = b_{k+1}$ for $k = i, \ldots, j-1$. In this sense, the set can be simplified as the following one:

$$\left\{ a_i, \underbrace{b_i, b_{i+1}, \ldots, b_{j-1}, b_j}_{b_k \ \textit{of all } T_{s_k}}, d_j \right\} \tag{2.23}$$

Step 2: To compute the four parameters (a, b, c, d) of $T(a, b, c, d)$:

(a) It is considered that the TrFN is used to represent a fuzzy envelope of the HFLTS. Since $\{a_i, b_i, b_{i+1}, \ldots, b_{j-1}, b_j, d_j\}$ is an ordered set, the parameters a and d of $T(a, b, c, d)$ can be easily computed respectively as the minimum and maximum elements of $\{a_i, b_i, b_{i+1}, \ldots, b_{j-1}, b_j, d_j\}$, i.e.,

$$\begin{cases} a = \min\{a_i, b_i, b_{i+1}, \ldots, b_{j-1}, b_j, d_j\} = a_i; \\ d = \max\{a_i, b_i, b_{i+1}, \ldots, b_{j-1}, b_j, d_j\} = d_j; \end{cases} \tag{2.24}$$

(b) The remaining elements $\{b_i, b_{i+1}, \ldots, b_{j-1}, b_j\}$ are then used to compute the parameters b and c of $T(a, b, c, d)$ using the OWA operator [83].

$$\begin{cases} b = \mathbf{OWA}_{\omega^s}(b_i, b_{i+1}, \ldots, b_{j-1}, b_j); \\ c = \mathbf{OWA}_{\omega^t}(b_i, b_{i+1}, \ldots, b_{j-1}, b_j). \end{cases} \tag{2.25}$$

being ω^s and ω^t two different weighting vectors (see [50] for more details).

Step 3: ω^s and ω^t are two exponential type of OWA weights, which were introduced to facilitate the computation of OWA weights (see [28] for further details).

Step 4: Finally, once the values of b and c are obtained, the fuzzy envelope of H_S can finally be denoted as

$$env_F(H_S) = T(a, b, c, d) \tag{2.26}$$

2.3.3 Extended comparative linguistic expressions with symbolic translation

Although the revised linguistic representation models and their extensions in the previous section model the uncertainty and hesitancy of information, they still lack interpretability of the results and loss information in the CW processes, which restrict their application to linguistic DM problems. Therefore, to overcome these shortcomings, the extended comparative linguistic expressions with symbolic translation (ELICIT) [62] model was developed to integrate the advantages of the 2-tuple linguistic representation model and CLE by extending the s_i of CLE to the 2-tuple linguistic values (s_i, α) to improve the accuracy and interpretability of the results. Similar to Definition 9, the ELICIT expressions are generated as follows.

Definition 11 *[62] Let G_H be a context-free grammar and $S = \{s_0, \dots s_g\}$ be a linguistic term set, and the elements of $G_H = (V_N, V_T, I, P)$ given by:*

$$
\begin{aligned}
V_N = \{ &(continuous\ primary\ term),\ (composite\ term),\ (unary\ relation),\\
&(binary\ relation), (conjunction)\}\\
V_T = \{ &at\ least,\ at\ most,\ between,\ and,\ (s_0,\alpha)^\gamma, (s_1,\alpha)^\gamma, \dots, (s_g,\alpha)^\gamma\}\\
I \in V_N &
\end{aligned}
$$

The production rules P defined in an extended Backus-Naur Form [7] are as follows:

$P = \{I\ ::=\ (continuous\ primary\ term)|(composite\ term)$

$(composite\ term)\ ::=\ (unary\ relation)(continuous\ primary\ term)|$

$(binary\ relation)(continuous\ primary\ term)(conjunction)(continuous\ primary\ term)$

$(continuous\ primary\ term)\ ::=\ (s_0,\alpha)^\gamma|(s_1,\alpha)^\gamma|\dots|(s_g,\alpha)^\gamma$

$(unary\ relation)\ ::=\ at\ least|at\ most$

$(binary\ relation)\ ::=\ between$

$(conjunction)\ ::=\ and\}$

where $\gamma \in \left[-\frac{1}{2g}, \frac{1}{2g}\right)$ is an additional parameter added after the CW scheme to avoid loss of information and to obtain accurate ELICIT results.

According to Definition 11, all possible ELICIT expressions [62] based on S are denoted as $ll_S^{\alpha,\gamma}$ and then provided as:

$$
ll_S^{\alpha,\gamma} = \begin{cases} (s_i,\alpha)^\gamma\,; \\ at\ most\ (s_i,\alpha)^\gamma\,; \\ at\ least\ (s_i,\alpha)^\gamma\,; \\ between\ (s_l,\alpha_l)^{\gamma_l}\ and\ (s_r,\alpha_r)^{\gamma_r}\,. \end{cases} \tag{2.27}
$$

Thus, in aiming to model the ELICIT expressions based on the CW scheme [80, 86, 94, 95] (shown in Figure 1.3), a FLA-based computational model, the ELICIT-CW scheme [62], is proposed. It is shown in Figure 2.7:

Before presenting more details about the ELICIT-CW scheme, two details should be clarified first:

(1) It can be seen that in the ELICIT-CW scheme, ELICIT information is handled by transforming it into TrFN [50, 89], the reason is because that any ELICIT

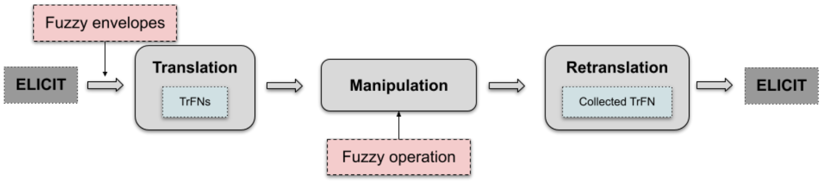

Figure 2.7: The ELICIT-CW scheme

value can logically be represented as TrFN, and TrFN is also sufficient in capturing and representing the uncertainty and fuzziness of such linguistic evaluations [19, 20].

Definition 12 *[50, 89] A TrFN is often indicated as* $T \equiv T(a,b,c,d)$ *with* $0 \leq a \leq b \leq c \leq d \leq 1$*, and is defined as a mapping* $\begin{array}{c} T : [0,1] \longrightarrow [0,1] \\ x \mapsto T(x) \end{array}$ *as follows:*

$$T(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq a; \\ \frac{x-a}{b-a} & \text{if } a < x < b \\ 1 & \text{if } b \leq x \leq c \\ \frac{d-x}{d-c} & \text{if } c < x < d \\ 0 & \text{if } d \leq x \leq 1. \end{cases} \tag{2.28}$$

For example, $T(0.2, 0.4, 0.6, 0.8)$ and $T(0.3, 0.7, 0.9)$ (see Figures 2.8 and 2.9).

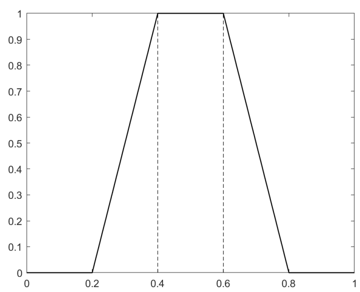

Figure 2.8: $T(0.2, 0.4, 0.6, 0.8)$

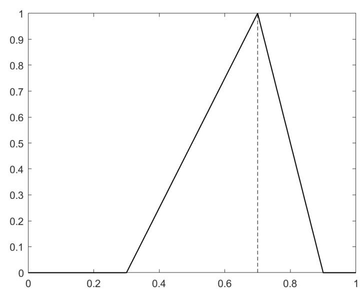

Figure 2.9: $T(0.3, 0.7, 0.9)$

Therefore, the set of all TrFNs defined on the unit interval $[0,1]$ is given as

$$\overline{T} = \{T(a,b,c,d) \mid T : [0,1] \longrightarrow [0,1] \;\; \textit{is a TrFN}\}. \tag{2.29}$$

(2) For the sake of clarity, these ELICIT expressions are simplified in [29] as follows:

$$\overline{\overline{S}} = \left\{ [(s_l, \alpha_l), (s_r, \alpha_r)]_{\gamma_l, \gamma_r} \;\middle|\; \forall (s_l, \alpha_l), (s_r, \alpha_r) \in \overline{S}, \;\; \textit{and } \gamma_l, \gamma_r \in \left[-\frac{1}{2g}, \frac{1}{2g}\right) \right\}. \tag{2.30}$$

Keeping in mind this clarification, the ELICIT-CW scheme is structured into the following three processes:

(1) **Translation process:** To transform ELICIT expressions into equivalent TrFNs by the definition of the fuzzy envelope function ζ^{-1}, as follows:

Definition 13 *[29] The fuzzy envelope of an ELICIT expression, also known as TrFNs, is defined as a mapping*

$$\zeta^{-1} : \overline{\overline{S}} \longrightarrow \overline{T}$$
$$[(s_l, \alpha_l), (s_r, \alpha_r)]_{\gamma_l, \gamma_r} \mapsto T(a, b, c, d)$$

and is computed as

$$\begin{aligned} a &= \gamma_l + \max\{\frac{\Delta_S^{-1}(s_l, \alpha_l) - \frac{1}{g}}{g}, 0\}, \quad b = \frac{\Delta_S^{-1}(s_l, \alpha_l)}{g}, \\ d &= \gamma_r + \min\{\frac{\Delta_S^{-1}(s_r, \alpha_r) + \frac{1}{g}}{g}, 1\}, \quad c = \frac{\Delta_S^{-1}(s_r, \alpha_r)}{g}. \end{aligned} \tag{2.31}$$

(2) **Manipulation process:** To carry out fuzzy aggregation, an aggregation operator is used to fuse the TrFNs and obtain a collective TrFNs.

(3) **Retranslation process:** To use the function ζ, namely the inverse of ζ^{-1}, and then retranslate the collective TrFN into its equivalent ELICIT value.

Definition 14 *[29] The function ζ is defined as a mapping, i.e.,*

$$\zeta : \overline{T} \longrightarrow \overline{\overline{S}}$$
$$T(a, b, c, d) \mapsto [(s_l, \alpha_l), (s_r, \alpha_r)]_{\gamma_l, \gamma_r}$$

and processed by

$$\begin{aligned} (s_l, \alpha_l) &= \Delta_S(gb) \quad \textit{with} \quad \gamma_l = a - \max\{b - \frac{1}{g^2}, 0\} \\ (s_r, \alpha_r) &= \Delta_S(gc) \quad \textit{with} \quad \gamma_r = d - \min\{c + \frac{1}{g^2}, 1\} \end{aligned} \tag{2.32}$$

In order to solve the linguistic decision problems, it is necessary to use aggregation operators in the selection process of the LDA. To this end, Labella et al. [62] presented fuzzy arithmetic mean operator to fuse such ELICIT values, as follows:

Definition 15 *[62] A fuzzy arithmetic mean (FAM) operator with dimension n that fuse ELICIT values is a mapping*

$$\Phi_{FAM} : \overline{\overline{S}}^n \longrightarrow \overline{\overline{S}}$$

such that

$$\Phi_{FAM}(x_1, x_2, \ldots, x_n) = \zeta\left(\sum_{i=1}^{n} \frac{1}{n} \cdot \zeta^{-1}(x_i)\right)$$
$$= \zeta\left(T\left(\frac{1}{n}\sum_{i=1}^{n} a_i, \frac{1}{n}\sum_{i=1}^{n} b_i, \frac{1}{n}\sum_{i=1}^{n} c_i, \frac{1}{n}\sum_{i=1}^{n} d_i\right)\right) \tag{2.33}$$

However, this operator does not take into account the relationships between aggregated ELICIT values. For this reason, with the aim of capturing different types of relationships between the ELICIT values, Dutta et al. [25] recently introduced the family of Bonferroni mean operators to fuse the ELICIT values. For the sake of space, here we only review the ELICIT Bonferroni mean (EBM) operator considering the homogeneous interrelation between ELICIT expressions (see [25] for further details), whose definition is given as:

Definition 16 *[25] For any $p, q \geq 0$ with $p + q > 0$, an ELICIT Bonferroni mean operator of dimension n is a mapping*

$$\Phi_{EBM} : \overline{\overline{S}}^n \longrightarrow \overline{\overline{S}}$$

and defined as:

$$\Phi_{EBM}(x_1, x_2, \ldots, x_n) = \zeta\left(\frac{1}{n(n-1)} \sum_{i,j=1; i\neq j}^{n} \zeta^{-1}(x_i)^p \otimes \zeta^{-1}(x_j)^q\right)^{\frac{1}{p+q}}$$
$$= \zeta(T(a, b, c, d))$$
$$\text{with} \begin{cases} a = \left(\frac{1}{n(n-1)} \sum_{i,j=1; i\neq j}^{n} a_i^p a_j^q\right)^{\frac{1}{p+q}}, \\ b = \left(\frac{1}{n(n-1)} \sum_{i,j=1; i\neq j}^{n} b_i^p b_j^q\right)^{\frac{1}{p+q}}, \\ c = \left(\frac{1}{n(n-1)} \sum_{i,j=1; i\neq j}^{n} c_i^p c_j^q\right)^{\frac{1}{p+q}}, \\ d = \left(\frac{1}{n(n-1)} \sum_{i,j=1; i\neq j}^{n} d_i^p d_j^q\right)^{\frac{1}{p+q}}. \end{cases} \tag{2.34}$$

2.4 Aggregation operators

As we know, the aggregation operator plays an important role in the selection process of decision analysis or LDA. Of the existing operators, the ordered weighted average (OWA) [78] operator and the induced OWA (IOWA) [87] operator have been widely used to fuse linguistic or numerical opinions, while the Choquet integral operator [14] or a combination with the OWA (or IOWA) operator has been widely used to fuse and describe the interrelationships among aggregation terms. Therefore, this section provides a short review of these operators, which would contribute to our understanding of the aggregation operators proposed in this research to deal with ELICIT values.

2.4.1 OWA operator

Due to the important contribution of the aggregation phase to the final decision [5, 30], a number of aggregation operators have been introduced in the literature to fuse the opinions of experts. Among them, the OWA aggregation operator, which was designed by Yager [78], stands out. It is defined as the fusion of numbers, as follows:

Definition 17 *[78] Let* $\mathbf{R}$ *be the set of all real numbers. An OWA operator is defined as fusing an n-dimensional vector and is defined as a mapping*

$$\begin{aligned}\mathbf{OWA}\ &: \mathbf{R}^n \longrightarrow \mathbf{R}\\ (x_1, x_2, \ldots, x_n) &\mapsto x\end{aligned}$$

that has an associated n-dimensional weighting vector $(\omega_1, \omega_2, \ldots, \omega_n)$ *to the order position satisfying the properties* $\omega_i \in [0,1]\,,\ \forall i \in \{1, \ldots, n\}$ *and* $\sum\limits_{i=1}^{n} \omega_i = 1$*, such that*

$$\mathbf{OWA}\,(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} \omega_j x_{\sigma(j)} \tag{2.35}$$

where a permutation function $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ *such that* $x_{\sigma(j)} \geq x_{\sigma(j+1)}$ *for all* $j \in \{1, \ldots, n-1\}$.

It can be seen that, for the application of the OWA operator, there are two issues to be solved. One is the order of the elements in the vector $(x_1, x_2, \ldots, x_n)$. The other is how to obtain the weights. Due to the inherent order of the real numbers, it remains to obtain the weights associated with the ordered positions according to the subjective opinion of the experts, which is a requirement for aggregation. Various methods are available to compute the weights of OWA [78, 79], including one of the most widely used methods presented by Yager, which is based on a regular monotone non-decreasing linguistic quantifier characterized by the membership function

$$Q_{\alpha,\beta}\left(r\right)\ :[0,1]\longrightarrow[0,1]$$

such that $Q_{\alpha,\beta}\left(r\right)$ is defined as a piecewise function as follows:

$$Q_{\alpha,\beta}\left(r\right)=\begin{cases} 0 & if \quad 0\leq r\leq\alpha\,; \\ \frac{r-\alpha}{\beta-\alpha} & if \quad \alpha<r<\beta\,; \\ 1 & if \quad \beta\leq r\leq 1\,. \end{cases} \tag{2.36}$$

where the two parameters α and β satisfy $0\leqslant\alpha<\beta\leqslant1$.

Due to the different values of α and β, there are several widely used linguistic quantifiers obtained as follows (see Figures 2.10, 2.11, 2.12, 2.13, 2.14 and 2.15):

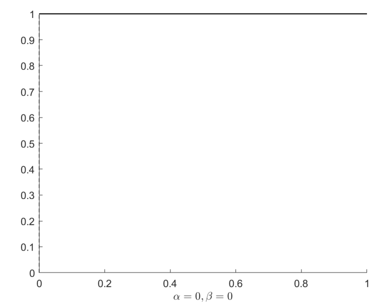

Figure 2.10: the linguistic quantifier "*for all*"

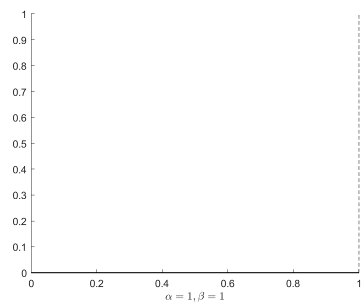

Figure 2.11: the linguistic quantifier "*there exist*"

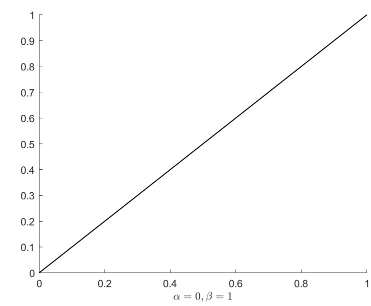

Figure 2.12: the linguistic quantifier "*mean*"

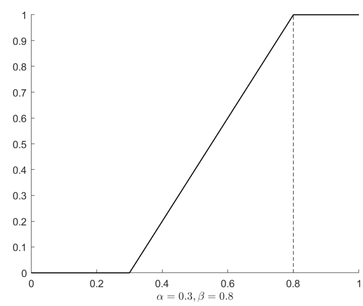

Figure 2.13: the linguistic quantifier "*most*"

And then the following formula computes weights ω_i of the OWA operator:

$$\omega_i=Q_{\alpha,\beta}\left(\frac{i}{n}\right)-Q_{\alpha,\beta}\left(\frac{i-1}{n}\right),\ i\in\{1,\ldots,n\} \tag{2.37}$$

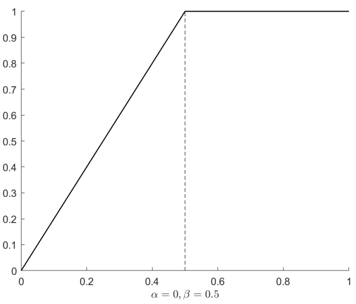

Figure 2.14: the linguistic quantifier "*at least half*"

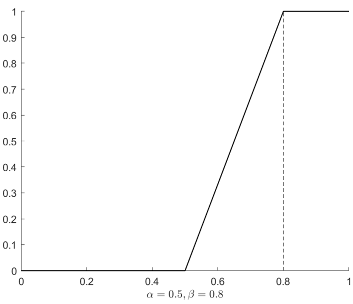

Figure 2.15: the linguistic quantifier "*as many as possible*"

These weights can be captured by two important concepts, one is *orness* [78] measure, the other one is *entropy* [78]. The former is used to determine the proximity of the operator to the maximum operator. It is defined as:

$$\begin{aligned} orness\,(\omega) &= \sum_{i=1}^{n} \omega_i \left(\frac{n-i}{n-1}\right) \\ &= \frac{1}{n-1} \sum_{i=1}^{n-1} Q_{\alpha,\beta}\left(\frac{i}{n}\right) \end{aligned} \tag{2.38}$$

Furthermore, if $n \to \infty$, then it is simplified as

$$orness\,(\omega) = \int_0^1 Q\,(r)\;dr \tag{2.39}$$

The latter is used to characterize how much information is used in the ω-based aggregation process, which is defined as:

$$Entropy\,(\omega) = -\sum_{i=1}^{n} \omega_i ln\,(\omega_i) \tag{2.40}$$

2.4.2 Type-1 OWA operator

To solve decision problems with fuzzy and imprecise opinions, several extensions of the OWA operator were introduced in [55, 56, 81, 97]. However, they all involve only the parameters of TrFN, whereas the weights are usually considered as numerical values. Although some of them use fuzzy weights, they lead to approximate results with some information loss due to computational rules of division. To avoid these, Zhou et al. [99] proposed a type-1 OWA operator that uses type-1 fuzzy set weights [23, 24] to fuse type-1 fuzzy numbers, which uses Zadeh's extension principle to define the type-1 OWA operator as follows:

Definition 18 *[99] Let $\mathcal{F}(\mathbf{X})$ be the set of type-1 fuzzy sets defined on the universe of discourse $\mathbf{X} \subseteq \mathbf{R}$, type-1 OWA operator is a mapping*

$$\Phi_{\mathbf{OWA}} \colon [\mathcal{F}(\mathbf{X})]^n \longrightarrow \mathcal{F}(\mathbf{X})$$
$$\left(\tilde{A}_1, \ldots, \tilde{A}_n\right) \longmapsto G$$

with an associated n-dimensional type-1 fuzzy weighting vector $(\tilde{\omega}_1, \ldots, \tilde{\omega}_n)$ to the order position whose membership function is defined as

$$\mu_{\tilde{\omega}_i}(\cdot) : [0,1] \longrightarrow [0,1]$$
$$\omega_i \mapsto \mu_{\tilde{\omega}_i}(\omega_i)$$

to fuse type-1 fuzzy numbers $\left(\tilde{A}_1, \ldots, \tilde{A}_n\right)$ by means of using the following formula:

$$\mu_G(y) = \sup_{\substack{\sum_{i=1}^n \overline{\omega}_i a_{\sigma(i)} = y \\ \omega_i \in [0,1], a_i \in \mathbf{X}}} \left(\mu_{\tilde{\omega}_1}(\omega_1) * \cdots * \mu_{\tilde{\omega}_n}(\omega_n) * \mu_{A_1}(a_1) * \cdots * \mu_{A_n}(a_n)\right) \tag{2.41}$$

where $*$ is a t norm operator, $\overline{\omega}_i = \frac{\omega_i}{\sum_{j=1}^n \omega_j}$, and $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ is a permutation function such that $a_{\sigma(i+1)} \leq a_{\sigma(i)}$ for all $i \in \{1, 2, \ldots, n-1\}$.

Remark 2 *Furthermore, assume that the associated weights are interval-valued fuzzy numbers $\hat{\omega}_i$ ($i = 1, 2, \ldots, n$), i.e., $\forall \alpha \in [0,1]$, the α-cuts of $\hat{\omega}_i$ are the same interval $\hat{\omega}_i^\alpha = \hat{\omega}_i^1 = \hat{\omega}_i^0$. To simplify the notation, the interval-valued fuzzy numbers $\hat{\omega}_i = \left[\omega_i^L, \omega_i^R\right]$ are noted as sub-interval of $[0,1]$. Let $\mathcal{I}([0,1])$ be the set of all subintervals of $[0,1]$, then $\forall\ \hat{\omega}_i \in \mathcal{I}([0,1])$ such that $\mu_{\hat{\omega}_i}(\omega) = \begin{cases} 1 & \text{if } \omega \in \hat{\omega}_i\,; \\ 0 & \text{otherwise}. \end{cases}$ Thus, Eq. (2.41) can be simplified as follows:*

$$\mu_G(y) = \sup_{\substack{\sum_{i=1}^n \overline{\omega}_i a_{\sigma(i)} = y \\ \omega_i \in \hat{\omega}_i^\alpha, a_i \in \mathbf{X}}} \left(\mu_{A_1}(a_1) * \cdots * \mu_{A_n}(a_n)\right) \tag{2.42}$$

It is easy to see that its interpretation and computation are still too complicated even using Eq. (2.42). Therefore, Zhou et al. [100] presented α-level type-1 OWA operators, which are defined to be equivalent to type-1 OWA operators based on α cuts of the fuzzy set [74, 82].

Definition 19 *[100] Let $\mathcal{F}(\mathbf{X})$ be the set of type-1 fuzzy sets defined in the discourse domain $\mathbf{X} \subseteq \mathbf{R}$. Given a n-dimensional type-1 fuzzy weighting vector $(\tilde{\omega}_1, \ldots, \tilde{\omega}_n)$, then for each $\alpha \in [0,1]$, an α-level type-1 OWA operator is a mapping*

$$\Phi_{\mathbf{OWA}}^\alpha \colon \left(\tilde{A}_1^\alpha, \ldots, \tilde{A}_n^\alpha\right) \longmapsto G^\alpha\,,$$

it is defined as

$$G^{\alpha} = \left\{ \frac{\sum_{i=1}^{n} \omega_i y_{\sigma(i)}}{\sum_{i=1}^{n} \omega_i} \middle| \forall \omega_i \in \tilde{\omega}_i^{\alpha}, y_i \in A_i^{\alpha} \right\} \tag{2.43}$$

where $\tilde{\omega}_i^{\alpha} = \{\omega \in [0,1] \mid \mu_{\tilde{\omega}_i}(\omega) \geqslant \alpha\}$, $\tilde{A}_i^{\alpha} = \left\{ y \in \mathbf{X} \middle| \mu_{\tilde{A}_i}(y) \geqslant \alpha \right\}$ *are two* α *cuts of their corresponding fuzzy numbers, respectively. And* $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ *is a permutation function such that* $y_{\sigma(i)} \geq y_{\sigma(i+1)}$ *for all* $i \in \{1, 2, \ldots, n-1\}$*.*

2.4.3 IOWA operator

When OWA operators are applied to fuse linguistic values or fuzzy numbers, it is necessary to provide a way to solve the issue of reordering argument variables whose order is not known or not easily available. A prominent approach is to add another variable, the so-called order-inducing variable u_i, to the argument variable x_i to form an OWA pair $\langle u_i, a_i \rangle$, $i \in \{1, \ldots, n\}$. Unlike OWA, the reordering step of the IOWA operator is determined by the order-inducing variable rather than by the argument variable itself, which is the essence of the induced OWA (IOWA) operator [87] proposed by Yager and Filev at that time. Its definition is as follows.

Definition 20 *[87, 85] Let* $\mathbf{R}$ *be the set of all real numbers and* $\mathbf{U}$ *be an ordered set. An IOWA operator is a mapping*

$$\mathbf{IOWA} : (\mathbf{U} \times \mathbf{R})^n \longrightarrow \mathbf{R}$$

that has an position-based weight vector $(\omega_1, \ldots, \omega_n)$ *verifying* $\omega_i \in [0,1]$, $\forall i \in \{1, 2, \ldots, n\}$ *and* $\sum_{i=1}^{n} \omega_j = 1$*, such that*

$$\mathbf{IOWA}(\langle u_1, x_1 \rangle, \ldots, \langle u_n, x_n \rangle) = \sum_{j=1}^{n} \omega_j x_{u-index(j)} \tag{2.44}$$

where $u - index(\cdot)$ *is an index function such that* $(x_1, \ldots, x_n)$ *is reordered according to the order-inducing variable* u_i *of the OWA pair* $\langle u_i, x_i \rangle$ *for all* $i \in \{1, 2, \ldots, n\}$*.*

2.4.4 Choquet integral operator

In MCDM problems, most problems are successfully solved under the assumption that the criteria are independent of each other. However, this is not always true in real-world problems. Therefore, the Choquet integral operator [14] based on fuzzy measures is a widespread and successful tool that has been used to model complex interactions between criteria and to capture the uncertainty inherent in the measurement of such relationships. To clarify the definition of the Choquet integral used in this paper, we will first recall Sugeno's fuzzy measure [57, 69] followed by the Choquet integral.

Definition 21 *[57, 69] Assume that* X *is a set and* $\mathcal{P}(X)$ *is the power set of* X *which is an* σ*- algebra, a fuzzy measure defined on* X *is a mapping* $\mu: \mathcal{P}(X) \longrightarrow [0,1]$*, such that*

(1) $\mu(\emptyset) = 0$ *and* $\mu(X) = 1$*;*

(2) If $A \subset B,\ \forall A, B \subset X$*, then* $\mu(A) \leq \mu(B)$*;*

(3) $\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\mu(A)\mu(B),\ \forall A, B \subset X$ *and* $A \cap B = \emptyset,\ \lambda \geqslant -1$*;*

(4) If $A_n \subset X,\ \forall n \in \mathbf{N}$ *and* $A_n \subset A_{n+1}$*, then* $\mu\left(\bigcup\limits_{n=1}^{\infty} A_n\right) = \lim\limits_{n\to\infty} \mu(A_n)$*;*

(5) If $A_n \subset X,\ \forall n \in \mathbf{N}$ *and* $A_{n+1} \subset A_n$*, then* $\mu\left(\bigcap\limits_{n=1}^{\infty} A_n\right) = \lim\limits_{n\to\infty} \mu(A_n)$*;*

Subsequently, the definition of the Choquet integral will be presented based on the pre-defined fuzzy measure.

Definition 22 *[31] Let* $X = (x_1, x_2, \ldots, x_n)$ *be a non-zero vector and* μ *be the fuzzy measure on* X*. The discrete Choquet integral of a function* $f: X \longrightarrow \mathbf{R}^+$ *with respect to* μ *is defined as*

$$(C) \int f d\mu = \sum_{i=1}^{n} f\left(x_{\sigma(i)}\right)\left[\mu\left(A_{\sigma(i)}\right) - \mu\left(A_{\sigma(i-1)}\right)\right] \tag{2.45}$$

where $\sigma: \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ *is a permutation function such that* $f\left(x_{\sigma(i-1)}\right) \geqslant f\left(x_{\sigma(i)}\right)$ *for all* $i \in \{1, \ldots, n\}$*.* $A_{\sigma(k)} \equiv \left(x_{\sigma(1)}, \ldots, x_{\sigma(k)}\right)$*, for all* $k \geqslant 1$*, and* $\mu\left(A_{\sigma(0)}\right) = 0,\ \mu\left(A_{\sigma(n)}\right) = 1$*.*

Chapter 3

Discussion of Results

In this chapter, the main proposals considered in this research are discussed. A brief discussion of the conclusions and findings of each proposal will be presented. It is structured into three proposals, which are divided into several parts.

(1) Fusing ELICIT expressions by means of aggregation operators. This proposal is divided into four parts:

  (a) A type-1 OWA operator for aggregating ELICIT expressions;

  (b) Induced OWA operator for GDM to deal with ELICIT expressions;

  (c) Application of Choquet integral operator to aggregate ELICIT expressions;

  (d) Application of Choquet-OWA operator to fuse ELICIT expressions.

(2) Definition of similarity measures. This proposal is divided into two parts.

  (a) Extend Tversky's ratio model to an asymmetric similarity measurement model with three conditional parameters;

  (b) Ranking of fuzzy numbers on the basis of new fuzzy distances;

(3) To introduce a new average consistency index based consensus model for a GDM dealing with ELICIT expressions.

3.1 Fusing ELICIT expressions by means of aggregation operators

As was aforementioned in Chapter 2.3.3, ELICIT is a novel linguistic representation model recently proposed to improve the interpretability and precision of the results by extending the concept of CLE to the continuous domain via the symbolic translation used in the 2-tuple linguistic model. Aggregation is the key to solving decision problems, but in the GDM problem, only two aggregation operators were defined to fuse ELICIT values [25, 62]. Therefore, it seems necessary to define new aggregation operators to model a wide range of decision scenarios. Afterward, we present several aggregation operators to fuse the ELICIT expressions.

3.1.1 A type-1 OWA operator for aggregating ELICIT expressions

One of the most widely used aggregation operators in decision making is the OWA operator. There are two key issues in applying the OWA operator: one is the reordering of the aggregated arguments variables, and the other is how to compute the weights.

(i) For the problem of reordering, ELICIT expressions do not have an inherent order and can be equivalently transformed into TrFN, then the ordering methods of TrFN can be used by the concept of *magnitude* introduced by Abbasbandy and Hajjari [1].

(ii) The weights are induced by the interval-valued linguistic quantifiers [99]. However, it is worth pointing out that the subtraction $J_{\frac{i}{n}} - J_{\frac{i-1}{n}}$ of $\hat{w}_i \equiv \left(J_{\frac{i}{n}} - J_{\frac{i-1}{n}}\right) \cap [0,1]$ for all $i \in \{1, \ldots, n\}$ introduced in [99] can not satisfy the *requisite equality constraints* condition [46]. In other words, if $J_{\frac{i}{n}} = J_{\frac{i-1}{n}}$, then $\hat{w}_i \neq 0$. For this reason, the weights of the newly proposed operator are slightly different compared to this one, and it is more reasonable to use the general interval subtraction method to prove the subtraction $J_{\frac{i}{n}} - J_{\frac{i-1}{n}}$ with the general interval subtraction $A \ominus B = \{a - b | \, \forall a \in A, \forall b \in B\}$ and

$$\hat{w}_i \triangleq \left(J_{\frac{i}{n}} \ominus J_{\frac{i-1}{n}}\right) \cap [0,1], \quad i \in \{1, \ldots, n\} \tag{3.1}$$

Therefore, based on these, a novel operator, namely ELICIT type-1 OWA (ELICIT-t1-OWA) operator, is proposed to perform the aggregation phase by extending the type-1 OWA [99, 100] to fuse the fuzzy envelopes of ELICIT information, i.e.,TrFNs, which includes the following:

(1) The ELICIT-t1-OWA operator is defined to aggregate the fuzzy envelopes of ELICIT expressions with interval weights.

(2) Furthermore, four classical properties of the newly proposed operator, such as idempotency, commutativity, monotonicity, and boundedness, are studied.

(3) An MCDM model is developed to deal with ELICIT values to solve such linguistic decision problems using the proposed ELICIT-t1-OWA operator, and its performance and effectiveness are demonstrated.

The article associated with this research is in Chapter 4.1, and is the following one:

Wen He, Rosa Mª Rodríguez, Bapi Dutta, Luis Martínez, A type-1 OWA operator for Extended Comparative Linguistic Expressions with Symbolic Translation, Fuzzy Sets and Systems, vol. 446, pp. 167-192, 2022. 10.1016/j.fss.2021.08.002

Furthermore, we continued studying the use of the type-1 OWA operator to aggregate ELICIT expressions and provided a definition of an aggregation operator called the type-1 ELICIT OWA operator, whose fuzzy weights are computed in the same way as proposed in [99], but differ from the ELICIT-t1-OWA operator.

This new definition was published in an International Conference introduced in Chapter 4.2, and is the following one:

Wen He, Rosa Mª Rodríguez, Bapi Dutta, Luis Martínez, Exploiting the type-1 OWA operator to fuse the ELICIT information, in IEEE CIS International Conference on Fuzzy Systems 2021, 11th-14th July, Luxembourg, 2021.

3.1.2 Induced OWA operator for GDM dealing with ELICIT expressions

In order to extend the classical IOWA operator to the aggregation of ELICIT expressions, we have defined two new aggregation operators, one of them is applied directly to the fusion of the ELICIT expressions and another is an extension of the type-1 OWA operator to a type-1 IOWA operator using Zadeh's extension principle [90, 91, 92]. These two aggregation operators are described as follows:

(1) According to Zadeh's extension principle [90, 91, 92], an IOWA operator is extended to fuse ELICIT expressions with crisp weights, the so-called ELICIT-IOWA operator. In this case, the weights are guided by regular monotonically nondecreasing linguistic quantifiers [28, 78, 79], such as *mean*, *at least half*, and *as many as possible*, etc.

(2) Similarly, the type-1 OWA operator will be extended as the induced type-1 OWA operator with fuzzy weights, the so-called t1-IOWA operator. Using Zadeh's extension principle, the t1-IOWA operator is also extended to fuse ELICIT

expressions, giving rise to the new ELICIT-t1-IOWA operator with interval weights induced by interval-valued linguistic quantifiers.

The article associated with this research is in Chapter 4.3, and is the following one:

Wen He, Bapi Dutta, Rosa Mª Rodríguez, Ahmad A. Alzahrani, Luis Martínez, Induced OWA Operator for Group Decision Making Dealing with Extended Comparative Linguistic Expressions with Symbolic Translation, Mathematics, vol. 9, n.º 20, 2021. 10.3390/math9010020

3.1.3 Application of Choquet integral operator to aggregate ELICIT expressions

As was aforementioned in section 2.1.3, a MCDM problem is composed of a set of finite alternatives $\mathcal{A} = \{\mathcal{A}_1, \cdots, \mathcal{A}_n\}$, $n \geq 2$, each of which can be assessed using a set of finite criteria $\mathcal{C} = \{\mathcal{C}_1, \cdots, \mathcal{C}_m\}$, $m \geq 2$. In most cases, all criteria are usually assumed to be independent to simplify the computational process, but we found correlations among these criteria, even among the alternatives of the GDM problems. Thus, the criteria for MCDM problems might be interdependent in real-world decision making problems. Bonferroni mean operator considers the interrelationship of the input argument variables and takes a direct link in the aggregated arguments. However, it cannot capture more complex interactions between criteria, such as positive and negative synergies between criteria. The Choquet integral [14] based on Sugeno fuzzy measures [68, 69] is a widely used tool to model such complex interactions between criteria and capture the uncertainty inherent in such relational measures. Therefore, the Choquet integral is applied to fuse ELICIT information to capture the interactions between criteria. The results obtained are the following ones:

(1) An average operator integrating the Choquet integral is proposed to fuse the ELICIT expressions, which is called the average-Choquet-ELICIT (AC-ELICIT) operator.

(2) A new MCDM model based on the AC-ELICIT operator is introduced to handle ELICIT expressions, and an illustrative example shows the practicality and feasibility of the AC-ELICIT operator.

The results of this research were published in the following international conference, which is in Chapter 4.4.

Wen He, Bapi Dutta, Rosa Mª Rodríguez, Luis Martínez, Application of Choquet Integral Operator to Aggregate ELICIT Information, in Intelligent and Fuzzy Systems, 24-26th August, Izmir (Turkey), vol. 307, pp. 272-280, 2021.

3.1.4 Application of Choquet-OWA operator to fuse ELICIT expressions

The weights of the OWA are usually obtained from linguistic quantifiers [84] under the assumption of complete independence between criteria, which may be unrealistic in some MCDM problems. In other words, the generation of the criteria weights used by the aggregation operator should take into account the different interrelationships between the criteria [2, 11] of each alternative. In this sense, the Choquet integral [10, 14] is able not only to capture but also to identify complex interactions between criteria by defining a predefined fuzzy measure. Therefore, we propose the ELICIT-Choquet-OWA operator, which combines the OWA and Choquet integral operators to fuse the ELICIT information. Thus, the results obtained in this proposal are the following.

(1) We define a new aggregation operator, the ELICIT-Choquet-OWA operator, which aggregates ELICIT values by considering the interrelationship between the criteria, combining the Choquet integral and the OWA operator;

(2) A brief study of the properties of the ELICIT-Choquet-OWA operator is presented;

(3) A novel MCDM model, which integrates the proposed aggregation operator to fuse the information and obtain the collective information, is developed.

The results of this research have been published in the following book chapter, which is in Chapter 4.5.

Wen He, Wei Liang, Álvaro Labella, Rosa Mª Rodríguez, Application of Choquet-OWA Aggregation Operator to Fuse ELICIT Information, in Advances in Complex Decision Making Using Machine Learning and Tools for Service-Oriented Computing, ISBN: 9781032375267, CRC Press (Accepted).

3.2 Similarity measures

Similarity, or distance measures, play a very pivotal role in real-world decision making problems and are widely used in face recognition [76], image recognition [49], anti-cheating [40], and recommendation systems [98], among others. Furthermore, in the implementation of CRPs, the calculation of the consensus level between experts necessarily involves the measurement of the distance, or rather the similarity between their preference values. Therefore, the definition and use of similarity/distance are worthy of in-depth study, especially in fuzzy or linguistic contexts.

3.2.1 Extend Tversky's ratio model to an asymmetric similarity measurement model with three conditional parameters

It is usual to measure and compute the similarity between objects in terms of the non-negative symmetric operator. For the sake of clarify, let $\mathbf{X}$ be a non-empty set and $\mathbf{R}^+$ be the set of all non-negative real numbers, the similarity measure is defined as a mapping $s : \mathbf{X} \times \mathbf{X} \longrightarrow \mathbf{R}^+$ such that $s(x, y)$ satisfies the following three general properties [66]:

(P1) **Equal self-similarity:** $\forall x, y \in \mathbf{X},\ s(x, x) = s(y, y) = 1;$

(P2) **Minimality:** $\forall x, y \in \mathbf{X},\ s(x, x) \ \geq \ s(x, y);$

(P3) **Symmetry:** $\forall x, y \in \mathbf{X},\ s(x, y) = s(y, x).$

However, in many cases, the symmetry property is not always true [3, 9, 27]. Furthermore, the ratio model originally proposed by Tversky [73] states that the similarity using feature matching methods is often asymmetric when $\gamma \neq \beta$ within γ, β are non-negative parameters associated with distinctive features of the compared objects. Moreover, in many practical situations, existing similarity measures using feature matching methods have some limitations:

(i) From a mathematical point of view, the assumption of feature-based evaluation logic, i.e., the use of binary features based on their values belonging to $\{0, 1\}$, will lead to unreliable results and limit their use, due to the fact that in most cases their actual values belong to $[0, 1]$.

(ii) Most of the existing asymmetric similarity methods integrate the Tversky's ratio model based on binary features with the classical vector-based similarity measure to avoid unreliable results due to ignoring the estimated values. It is certainly a simple task to compute recommendations for some features when using this similarity measure in the recommender system [59]. However, when the number of features increases to hundreds or even thousands, the computation becomes more complex and unintuitive, making the operation a challenge.

(iii) Tversky's ratio model uses the value of 1 instead of another parameter to describe common features, which results in a final similarity that does not reflect the significant or negligible impact of common features. Therefore, experts may find it more intuitive to use three parameters to better account for the importance of common and different features.

Considering these limitations, we extend Tversky's ratio model by adding another parameter characterizing common features and propose an asymmetric similarity measure that includes the following main contributions:

(1) A new parameter, α, is added to Tversky's ratio model to characterize common features and meets the condition: (i) $\alpha \in (0,1], \gamma, \beta \in [0,1]$; (ii) $\gamma \neq \beta$, and (iii) $\gamma + \alpha + \beta = 1$. Therefore, a new extended model called the three-parameter asymmetric similarity measures (3p-ASM) model is introduced.

(2) Due to the estimated values of features being not only in a binary set $\{0,1\}$, but also in $[0,1]$, the 3p-ASM is reformulated on the basis of set theories (crisp and fuzzy sets) and vector theory.

(3) The properties of different forms of 3p-ASM models are analyzed to obtain a wider applicability.

The article associated with this proposal is in Chapter 4.6, and is the following one.

Wen He, Bapi Dutta, Yaya Liu, Rosa Mª Rodríguez, Extend Tversky's Ratio Model to an Asymmetric Similarity Measurement Model with Three Conditional Parameters: 3p-ASM Model, International Journal of Computational Intelligence Systems, vol. 16, 113, 2023, 10.1007/s44196-023-00285-8.

3.2.2 Ranking of fuzzy numbers on the basis of new fuzzy distances

Due to the fact that ranking fuzzy numbers is a required and complex task in several steps of the decision making process under uncertainty, different methods of ranking fuzzy numbers have been studied [75]:

(i) Methods based on defuzzification;

(ii) Methods based on the distance between fuzzy numbers;

(iii) Pairwise comparison methods.

To facilitate the ranking of fuzzy numbers, we highlight useful methods for constructing fuzzy distances from classical interval distances, since their α cut is a continuous interval. Existing interval distances are usually computed using the endpoints or midpoints of the interval, and the results may not reflect the correct distance due to missing information. To avoid this, all points belonging to the two corresponding intervals should be involved in the calculation of the interval distance, which can be defined as the average of the distances between any two points belonging to the two corresponding intervals. Therefore, the following is a short list of the main contributions:

(1) A novel interval distance is defined by using the concept of integral, in the sense that it computes the distance as the average of the sum of all distances between any two points in two intervals, respectively. This means that the more information used to reflect the distance, the more correct and reliable the result will be.

(2) On the basis of the previous new interval distances, new distances between fuzzy numbers are presented and their properties are proved.

(3) Then, new fuzzy number ranking indices are naturally introduced based on the new fuzzy distances between each fuzzy number and the ideal fuzzy number.

(4) To demonstrate the advantages of the novel proposed interval distance, fuzzy distance, and ranking methods, several numerical analyses and comparisons are presented.

The article associated with this proposal is in Chapter 4.6, and is the following one.

Wen He, Rosa Mª Rodríguez, Zdenko Takáč, Luis Martínez, Ranking of fuzzy numbers on the basis of new fuzzy distance, International Journal of Fuzzy Systems (accepted), 10.1007/s40815-023-01571-5.

3.3 Average consistency index based consensus model for a group decision making problem dealing with ELICIT expressions

Consistency and consensus [21, 36] are the two main different measures involved in the study when using the RPR structure for GDM problems. Taking into account the advantages of ELICIT expressions to improve the accuracy of computational results, a new linguistic RPR structure using ELICIT information, named ELICIT-RPR, is proposed. Based on the ELICIT-CW scheme, Labella et al. [47] proposed an initial consensus model for ELICIT-RPR based on the Manhattan distance between TrFNs, and Hua et al. [39] introduced a consensus model for a social network using ELICIT-RPR based on the Euclidean distance between TrFNs. Both of them actually deal with TrFN-RPRs and provide the same solution process as shown in Figure 3.1.

However, none of them considered the consistency index of ELICIT-RPR, which may lead to unreliable results because inconsistency may imply irrational random preferences. Therefore, we propose a new consensus model based on the ELICIT-RPRs' consistency index. To simplify computation within the mathematical frame-

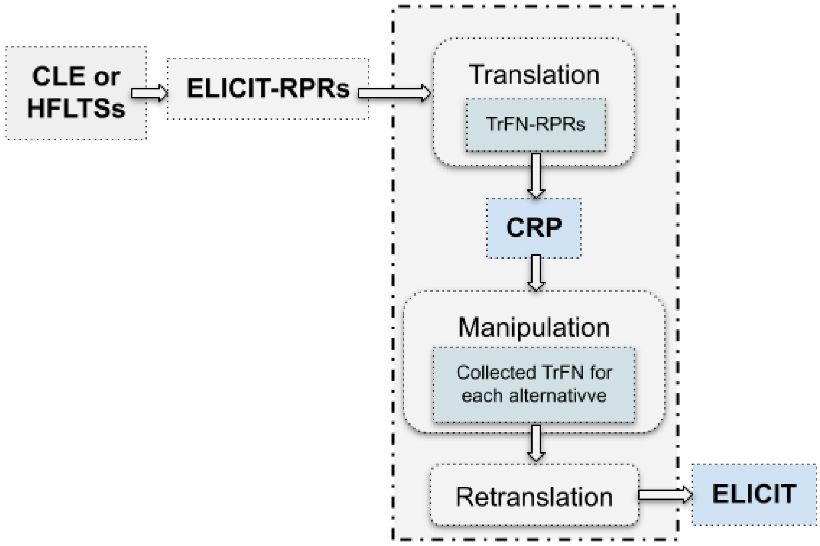

Figure 3.1: The solving process of GDM dealing with ELICIT information

work and to facilitate computation between ELICIT expressions, the elements of ELICIT-RPR are unified into a 2-tuple linguistic interval [29]. Then, the new GDM solution process considering consensus and consistency is shown as Figure 3.2.

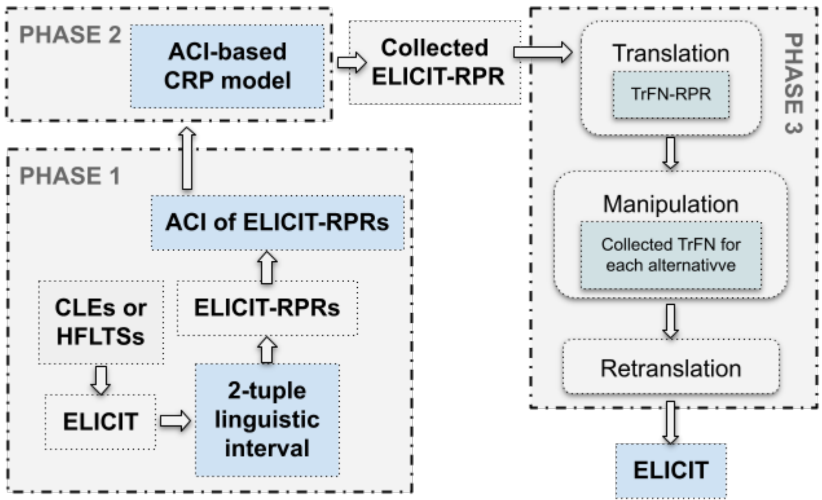

Figure 3.2: The new solving process of GDM dealing with ELICIT information

The novel GDM solving process is divided into 3 main phases, as Figure 3.2 shows.

Phase 1: A new L-RPR structure is introduced to deal with ELICIT information by a linguistic interval format. Then, the definition of the average consistency index (ACI) of ELICIT-RPR is proposed.

Phase 2: The novel ACI-based consensus model is defined. It is divided into 3 steps as follows:

Step 1: An algorithm to improve the unacceptable consistency is then presented.

Step 2: A novel global consensus index is computed on the basis of a new distance between the two ELICIT-RPRs.

Step 3: A new consensus model to deal with ELICIT-RPRs based on their ACI to achieve higher agreement among a group of experts of the GDM problem is proposed.

Phase 3: To complete the solution scheme of ELICIT-DA, the ELICIT-CW scheme should be applied in three steps:

Step 1: *Translation process:* The fuzzy envelopes of ELICIT expressions are obtained as TrFNs by applying the function ζ^{-1}, i.e., Eq. (2.31), the ELICIT-RPR is the matrix $\tilde{\mathcal{M}}$ as follows:

$$\tilde{\mathcal{M}} = \begin{pmatrix} T_{11} & \cdots & T_{1j} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ T_{i1} & \cdots & T_{ij} & \cdots & T_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nj} & \cdots & T_{nn} \end{pmatrix} \tag{3.2}$$

Step 2: *Manipulation process:* It is actually an aggregation process, in which we will select an appropriate aggregation operator and apply it to fuse the TrFNs for each alternative. The collective TrFNs vector $\mathbf{T}_c$ is then obtained as follows:

$$\mathbf{T}_c = \begin{pmatrix} T_{1,c} \\ \vdots \\ T_{i,c} \\ \vdots \\ T_{n,c} \end{pmatrix} \tag{3.3}$$

Step 3: *Retranslation process:* To generate results that are understandable and straightforward, the function ζ, i.e., Eq. (2.31), transforms the collective TrFNs into ELICIT expressions.

The article associated with this proposal is in Chapter 4.8.

Wen He, Rosa Mª Rodríguez, Luis Martínez, Average consistency index based consensus model for a group decision making problem dealing with ELICIT expressions, Computers & Industrial Engineering (Under review in second round).

Chapter 4

Publications

By virtue of the provisions of article 25, point 2, of the current regulations for Doctoral Studies at the University of Jaén, corresponding to the RD program. 99/2011, this chapter presents the publications that make up the core of this doctoral thesis.

These publications correspond to three scientific articles published, one accepted, and another submitted to International Journals indexed by the JCR (Journal Citation Reports) database, produced by Clarivate Analytics, a book chapter published in CRC Press, and two international conference contributions.

4.1 A type-1 OWA operator for Extended Comparative Linguistic Expressions with Symbolic Translation

- State: Published.
- Title: A type-1 OWA operator for Extended Comparative Linguistic Expressions with Symbolic Translation.
- Authors: Wen He, Rosa Mª Rodríguez, Bapi Dutta and Luis Martínez.
- Journal: Fuzzy Sets and Systems.
- Volume: 446. Page: 167-192. Date: 5 October 2022.
- DOI: https://doi.org/10.1016/j.fss.2021.08.002
- ISSN: 0165-0114.
- Impact factor (JCR 2021): 4.462.
  - Quartiles:
    * Quartile 1 COMPUTER SCIENCE, THEORY & METHODS 19/110.
    * Quartile 1 MATHEMATICS, APPLIED 6/267.

4.2 Exploiting the type-1 OWA operator to fuse the ELICIT information

- State: Published.
- Title: Exploiting the type-1 OWA operator to fuse the ELICIT information.
- Authors: Wen He, Rosa Mª Rodríguez, Bapi Dutta and Luis Martínez.
- Conference: 2021 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE).
- DOI: https://ieeexplore.ieee.org/document/9494400
  - Quartiles:
    - Ranking Core: Core A.

4.3 Induced OWA operator for Group Decision Making dealing with Extended Comparative Linguistic Expressions with Symbolic Translation

- State: Published.
- Title: Induced OWA operator for Group Decision Making dealing with Extended Comparative Linguistic Expressions with Symbolic Translation.
- Authors: Wen He, Bapi Dutta, Rosa Mª Rodríguez, Ahmad A. Alzahrani and Luis Martínez.
- Journal: Mathematics.
- Volume: 9(20). Date: 23 December 2020.
- DOI: https://doi.org/10.3390/math9010020.
- ISSN: 2227-7390.
- Impact Factor (JCR 2021): 2.592.
  - Quartiles:
    * Quartile 1 in MATHEMATICS 21/333.

4.4 Application of Choquet integral operator to aggregate ELICIT information

- State: Published.
- Title: Application of Choquet integral operator to aggregate ELICIT information.
- Authors: Wen He, Rosa Mª Rodríguez, Bapi Dutta and Luis Martínez.
- Conference: 3rd Intelligent and Fuzzy Systems Conference (INFUS 2021).
  - Lecture Notes in Networks and Systems book series.
- DOI: https://link.springer.com/chapter/10.1007/978-3-030-85626-7_33

4.5 Application of Choquet-OWA Aggregation Operator to Fuse ELICIT Information

- State: Accepted (Book Chapter).
- Book chapter: Application of Choquet-OWA Aggregation Operator to Fuse ELICIT Information.
- Book: Advances in Complex Decision Making Using Machine Learning and Tools for Service-Oriented Computing
- Authors: Wen He, Wei Liang, Álvaro Labella and Rosa Mª Rodríguez.
- Editorial: CRC Press.
- ISBN: 9781032375267.

4.6 Extend Tversky's Ratio Model to an Asymmetric Similarity Measurement Model with Three Conditional Parameters: 3p-ASM Model

- State: Accepted.
- Title: Extend Tversky's Ratio Model to an Asymmetric Similarity Measurement Model with Three Conditional Parameters: 3p-ASM Model
- Authors: Wen He, Bapi Dutta, Yaya Liu and Rosa Mª Rodríguez.
- Journal: International Journal of Computational Intelligence Systems.
- Volume: 16. Page: nº 113. Date: 11 July 2023.
- DOI: https://doi.org/10.1007/s44196-023-00285-8
- ISSN: 1875-6883.
- Impact Factor (JCR 2021): 2.259.
  - Quartiles:
    * Quartile 3 COMPUTER SCIENCE, ARTIFICIAL INTELLIGENCE 124/190.
    * Quartile 3 COMPUTER SCIENCE, INTERDISCIPLINARY APPLICATIONS 106/156.

4.7 Ranking of fuzzy numbers on the basis of new fuzzy distance

- State: Accepted.
- Title: Ranking of fuzzy numbers on the basis of new fuzzy distance.
- Authors: Wen He, Rosa Mª Rodríguez, Zdenko Takáč and Luis Martínez.
- Journal: International Journal of Fuzzy Systems.
- DOI: 10.1007/s40815-023-01571-5
- Impact factor (JCR 2021): 4.085.
  - Quartiles:
    - Quartile 2 AUTOMATION & CONTROL SYSTEMS 23/65.
    - Quartile 2 COMPUTER SCIENCE, ARTIFICIAL INTELLIGENCE 61/145.

4.8 Average consistency index based consensus model for a group decision making problem dealing with ELICIT expressions

- State: Under review in the second round.
- Title: Average consistency index based consensus model for a group decision making problem dealing with ELICIT expressions.
- Authors: Wen He, Rosa Mª Rodríguez and Luis Martínez.
- Journal: Computers & Industrial Engineering.
- Impact factor (JCR 2021): 7.180.
  - Quartiles:
    - Quartile 1 COMPUTER SCIENCE, INTERDISCIPLINARY APPLICATIONS 19/112.
    - Quartile 1 ENGINEERING, INDUSTRIAL 12/50.

Chapter 5

Conclusions and Future Works

Finally, this chapter summarizes the current research memory, reviews the main proposals and results obtained across this memory, and suggests future work.

5.1 Conclusions

Decision making is a common human process in many real-world activities, such as engineering, medicine, etc. Real-world decision problems are usually defined in contexts where the information is fuzzy and imprecise, and therefore there are different approaches to deal with this type of uncertainty. In this research memory, we focus on the fuzzy linguistic approach that provides successful results in decision making problems. The use of fuzzy linguistic approach implies computing with words (CW) processes. In the literature, there are different computational linguistic models to model the linguistic information, however, most of them use linguistic terms defined a priori, and limit experts to eliciting their opinions using a single linguistic term. Therefore, it was pointed out that it is necessary to improve the flexibility of expressing more complex linguistic terms, due to the fact that sometimes the use of a single linguistic term is not sufficient to reflect the expert's knowledge. To overcome these limitations, the use of context-free grammars was introduced to generate comparative linguistic expressions that are close to human modeling and capable of simulating the hesitations of experts. These linguistic expressions are based on the concept of hesitant fuzzy linguistic term set. However, the loss of information in the CW process and the interpretability of its results limit the use of the hesitant fuzzy linguistic term sets. Therefore, a new representation model, so-called extended comparative linguistic expressions with symbolic translation (ELICIT), has been recently introduced together with a computational model. This novel linguistic model improves the accuracy of the results because the CW process using symbolic translation has no approximation and is capable of maintaining the interpretability of the results. This model was proposed to deal with decision problems, so it is necessary to define novel aggregation operators, similarity measures, group decision models, consensus models, etc., which are capable of managing ELICIT expressions. Thus, we have obtained the following results: expressions

(1) The aggregation process plays a crucial role in the solution scheme of a decision making problem; therefore, several new aggregation operators have been proposed in this research memory to model a wide range of decision scenarios.

    (a) Using Zadeh's extension principle, the type-1 OWA operator has been extended to define two aggregation operators that fuse ELICIT expressions, one of which is type-1 ELICIT-OWA operator. The other, called the ELICIT-t1-OWA operator, has used novel interval weights induced by new interval-valued linguistic quantifiers.

    (b) Similarly, the induced OWA operator has been extended to the type-1 IOWA operator to fuse ELICIT expressions. It is denoted as the ELICIT-t1-IOWA operator.

    (c) Choquet integral is usually used to capture the interrelations between criteria in multi-criteria decision making problems; then, several types of Choquet integral operators such as the Choquet-OWA operator and the average Choquet operator have been proposed to fuse ELICIT expressions.

(2) Ranking ELICIT expressions is a necessary and complex task, and the challenge is to compute the similarity or distance between two ELICIT expressions. Taking into account that ELICIT expressions are usually handled by their equivalent fuzzy envelope, i.e., trapezoidal fuzzy numbers, we have extended Tversky's ratio model in the form of crisp sets, fuzzy sets and vectors to develop the measure of asymmetric similarity. We have also introduced a novel distance between fuzzy numbers.

(3) Finally, we have also defined a consensus model for ELICIT expressions that takes into account the consistency of experts' preferences to obtain reliable and easy-to-understand results.

Accordingly, it should be noted that all the objectives identified at the beginning of this research memory have been accomplished, providing tools, models, and results that improve on the state-of-the-art prior to our research and offer the possibility of new research, as described in the following sections.

5.2 Future Works

From the results obtained in this research, it is possible to identify some possible proposals to continue with the research carried out in this doctoral thesis. These future works are the following:

(1) To propose new methods for computing the envelopes of ELICIT expressions and then, on the basis of these new methods, novel aggregation operators and distance measures shall be proposed.

(2) Some researchers have pointed out that words mean different things to different people and have introduced several proposals for decision making using personalized individual semantics. It seems very interesting to use personalized individual semantics in consensus models dealing with ELICIT expressions.

(3) Sometimes there are decision making problems where experts involved have different knowledge or backgrounds, and the use of multigranular linguistic models is convenient. Since, these models could be extended to use ELICIT expressions.

(4) To define decision making models that use heterogeneous information, including ELICIT expressions.

5.3 Additional publications

Other publications related to this research memory are the following ones:

International Conferences

(1) Wen He, Rosa M. Rodríguez, Bapi Dutta, Luis Martínez, Aggregation of Comparative Linguistic Expressions with Symbolic Translation by IOWA Operator for Group Decision Making, in International Virtual Workshop on Business Analytics Eureka, 2-4th June, Ciudad Juarez (México), 2021.

(2) Wen He, Rosa M. Rodríguez, Luis Martínez, Asymmetric distance-based Comprehensive Minimum Cost Consensus Model, in 15th International FLINS Conference on Machine Learning, Multi agent and Cyber physical systems and the 17th International ISKE Conference (FLINS/ISKE 2022) Tianjin (China) 26-28 August, 2022.

5.4 Research stays

During the development of this Ph.D. thesis, several stays and collaborations abroad have taken place with the aim of improving the research training of Ph.D. students through the knowledge and experience of experts.

Thanks to the EDUJA 2022 fellowship from the University of Jaén, I was able to spend 3 months at the Department of Mathematics of the Slovak University of Technology in Bratislava, Slovakia, working closely with Prof. Zdenko Takáč.

In addition, I have the opportunity to spend a month at the Universidad Pública de Navarra (Spain) with Prof. Humberto Bustince Sola supported with a research project FEDER-UJA of the research group SINBAD[2].

Appendix A

Resumen escrito en Español

Título de la tesis: *Operaciones para expresiones lingüísticas complejas para toma de decisiones bajo incertidumbre.*

Este apéndice incluye el título, índice, introducción, resumen y conclusiones escritas en español como parte de los requisitos necesarios para obtener el doctorado según el artículo 23.2 del Reglamento de Estudios de Doctorado de la Universidad de Jaén.

En primer lugar, se muestra el índice de esta memoria de investigación. A continuación se introduce brevemente la investigación llevada a cabo, indicando la motivación, objetivos planteados y la estructura en capítulos que componen esta tesis. Se presenta también un resumen de la misma, y finalmente se describen las conclusiones obtenidas y trabajos futuros.

Contenidos

A.1 Introducción

A.1.1 Motivación

A la hora de resolver problemas reales de toma de decisiones en grupo (TDG), el análisis de decisiones clásico [63] tiene como objetivo encontrar la mejor solución entre un conjunto de alternativas basándose en las opiniones expresadas por un conjunto de expertos. La solución del análisis de decisiones clásico consta básicamente de dos fases (véase la figura A.1) [63]:

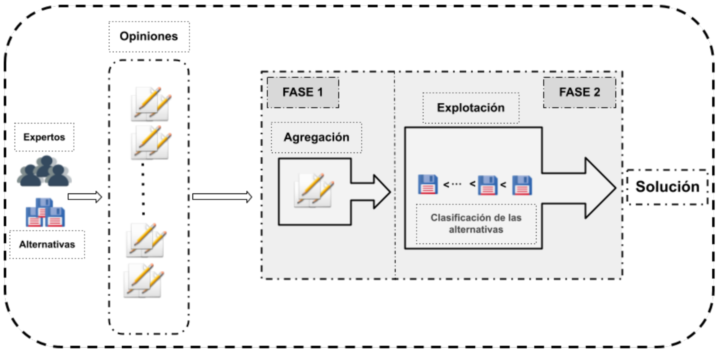

Figure A.1: Esquema general del análisis de decisiones clásico

(1) **Fase de agregación:** Las opiniones proporcionadas por los expertos son agregadas para obtener una opinión colectiva.

(2) **Fase de explotación:** A partir de la opinión colectiva obtenida en la fase de agregación anterior, se obtiene un ranking, clasificación o selección entre las alternativas y se selecciona la mejor alternativa como solución al problema.

Sin embargo, en problemas de TDG definidos en contextos cualitativos donde los expertos no pueden expresar sus opiniones numéricamente, resulta útil utilizar variables lingüísticas. En estas situaciones, el análisis de decisiones clásico puede ampliarse al análisis de decisiones lingüístico (ADL), que puede utilizarse para resolver problemas de TDG en casos en los que los expertos utilizan variables lingüísticas para expresar sus opiniones. Así, la solución para el ADL consta de los tres siguientes pasos (véase la figura A.2) [33]:

Paso 1: Seleccionar un conjunto de términos lingüísticos con la granularidad, sintaxis y semántica adecuadas [34], que utilizarán los expertos para expresar sus opiniones.

Paso 2: Elegir un operador de agregación adecuado para fusionar las opiniones lingüísticas proporcionadas.

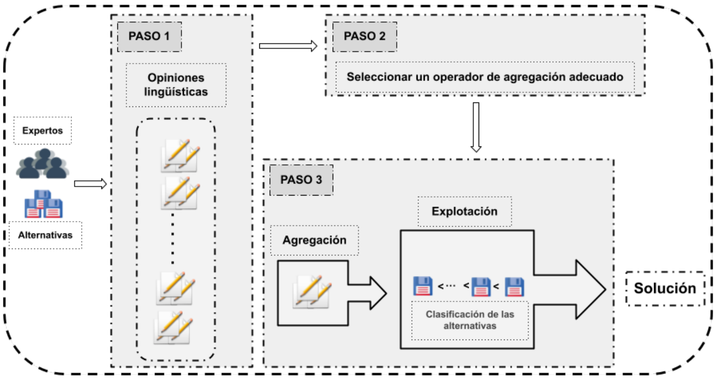

Figure A.2: Esquema general de solución del análisis de decisión lingüístico

Paso 3: Se incluye el análisis de decisiones clásico, pero adaptado para manejar la información lingüística en las dos fases siguientes:

(a) **Fase de agregación:** Las opiniones lingüísticas proporcionadas por los expertos son agregadas utilizando un operador de agregación seleccionado para obtener una opinión lingüística colectiva.

(b) **Fase de explotación:** A partir de la opinión lingüística colectiva, se obtiene un orden de clasificación entre las alternativas para seleccionar la mejor.

En el paso 1 del ADL, una vez predeterminado el conjunto de términos lingüísticos, se pueden utilizar variables lingüísticas para obtener las opiniones lingüísticas de los expertos. Debido a la incertidumbre inherente en las variables lingüísticas, el uso de enfoques lingüísticos difusos (ELD) [90, 91, 92], es decir, técnicas de aproximación basadas en la teoría de conjuntos difusos [96], es necesario e imperativo. En la literatura, existen varios modelos lingüísticos computacionales basados en el ELD para modelar la incertidumbre que siguen el esquema de computación con palabras (CP) [80, 86, 94, 95] (ver Figura A.3), señalando la importancia de los procesos de traducción y retraducción en CP. Por ejemplo, Herrera y Martínez [35] introdujeron

Figure A.3: Esquema de CP

el modelo lingüístico 2-tupla para obtener resultados precisos mediante el concepto de traslación simbólica y manteniendo la interpretabilidad de los resultados. Sin embargo, este modelo tiene una limitación, porque utiliza términos lingüísticos simples para representar el conocimiento de los expertos, y a veces el uso de un único término lingüístico no es suficiente. Para superar esta limitación, algunos enfoques han intentado generar expresiones lingüísticas más flexibles y ricas que los términos lingüísticos simples, pero la mayoría de estos modelos proporcionan expresiones lingüísticas que distan mucho del razonamiento humano o no definen formalmente cómo establecer dichas expresiones. Por ello, Rodríguez et al. [61] introdujeron el uso de una gramática libre de contexto para generar expresiones lingüísticas comparativas (ELC) que siguen el ELD y se basan en el concepto de conjuntos de términos lingüísticos difusos dudosos (DTLDD) [60], que son capaces de modelar las opiniones proporcionadas por los expertos cuando dudan entre múltiples términos lingüísticos. Sin embargo, la interpretabilidad de los resultados de los modelos computacionales existentes y sus extensiones, así como la pérdida de información en los procesos de CP, pueden limitar su uso. Para abordar estos inconvenientes, recientemente se ha propuesto un nuevo modelo de representación lingüístico difuso para ELC, denominado expresiones lingüística comparativa extendida con traslación simbólica (ELICIT) [62], junto con un modelo computacional. Este modelo lingüístico ELICIT mejora la precisión de los resultados al extender la representación de las ELC a un dominio continuo, utilizando el concepto de traslación simbólica, sin realizar ninguna aproximación en los procesos de CP y manteniendo la comprensión e interpretabilidad de los resultados. Por lo tanto, es beneficioso utilizar el nuevo modelo lingüístico ELICIT para que los expertos proporcionen sus opiniones en comparación con otros modelos lingüísticos existentes. De esta forma, el ADL se redefine como ELICIT-AD, como se muestra en la Figura A.4:

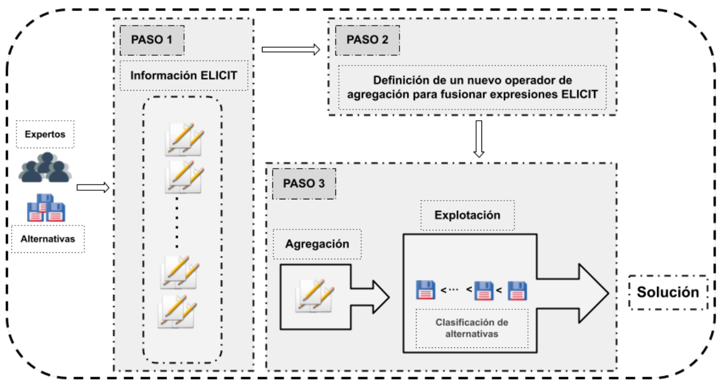

Figure A.4: Esquema de solución de ELICIT-AD

Obviamente, el segundo paso de ELICIT-AD requiere la selección de operadores de agregación para fusionar expresiones ELICIT, que pueden aplicarse para resolver problemas de TDG realizando el tercer paso de ELICIT-AD. Según la investigación actual sobre los operadores de agregación con diferentes características, tales como el operador de agregación de la media aritmética, operador de agregación de la media ponderada, operador de la media ponderada ordenada (OWA) [83], y el operador de agregación de la integral de Choquet [14, 31] son más prominentes y ampliamente utilizados en los esquemas clásicos de solución de análisis de decisión. Por lo tanto, estos operadores de agregación también pueden aplicarse para agregar expresiones ELICIT, lo que naturalmente conduce a la definición de nuevos operadores de agregación para expresiones ELICIT.

Sin embargo, los resultados colectivos obtenidos podrían no ser aceptados por todos los expertos [36]. En otras palabras, algunos expertos consideran que sus opiniones no se han tenido en cuenta. Por lo tanto, antes del tercer paso de ELICIT-AD es necesario añadir un proceso de alcance de consenso (PAC). Basándonos en la investigación actual sobre los PAC, parece necesario estudiar nuevos retos para aplicar PAC con expresiones ELICIT. Otro reto de los PAC es estudiar la medida de distancia, ya que todos los modelos de consenso utilizan medidas de distancia simétricas, sin embargo, parece interesante analizar el impacto de la distancia asimétrica en PAC.

A.1.2 Objetivos

Partiendo de la motivación y consideraciones expuestas en el apartado anterior, el objetivo de esta tesis doctoral se centra en la investigación y definición de nuevos operadores de agregación y modelos de decisión para manejar expresiones ELICIT en problemas de TDG. Teniendo en cuenta este propósito, nos planteamos los siguientes objetivos:

(1) Dado que el modelo lingüístico ELICIT ha sido propuesto recientemente, existen pocos operadores de agregación para fusionar expresiones ELICIT. Debido a la importancia del uso de operadores de agregación en el esquema de solución ELICIT-AD, es necesario proponer nuevos operadores difusos de agregación para fusionar expresiones ELICIT sin pérdida de información y con el objetivo de obtener resultados precisos y fácilmente comprensibles.

(2) Los modelos de toma de decisiones multicriterio (TDMC) se utilizan ampliamente para resolver problemas de TDG cuando se utilizan múltiples criterios en el proceso de decisión. Por lo tanto, se propone un nuevo modelo difuso de TDMC para seleccionar la(s) mejor(es) alternativa(s) como solución a este problema mediante la agregación de expresiones ELICIT para cada alternativa evaluada respecto a los criterios, a través de un operador de agregación predefinido.

(3) Basándose en el hecho de que las expresiones ELICIT y los números difusos

trapezoidales pueden transformarse de forma equivalente mediante envelops difusos y teniendo en cuenta que las medidas de distancia/similitud se utilizan a menudo para definir medidas de consenso u órdenes de clasificación, se proponen nuevas medidas de distancia/similitud y de distancia/similitud asimétricas para las expresiones ELICIT.

(4) Basándose en las nuevas medidas de distancia/similitud, se define un nuevo modelo de consenso para expresiones ELICIT y se estudia su funcionamiento. Algunos modelos de consenso obtienen resultados sesgados porque no consideran la consistencia de las preferencias proporcionadas por los expertos [8, 13, 38, 48, 77], por tanto, un nuevo modelo de consenso que tiene en cuenta la consistencia de las preferencias de los expertos para manejar expresiones ELICIT en problemas TDG será definido.

A.1.3 Estructura

Para la consecución de los objetivos presentados en el apartado anterior, teniendo en cuenta el artículo 23, punto 3, de la normativa vigente de Estudios de Doctorado de la Universidad de Jaén, de acuerdo con el programa establecido en el RD 99/2011, esta memoria de investigación se presentará como un conjunto de artículos publicados por el doctorando.

Se han publicado dos artículos en revistas internacionales incluidas en la base de datos JCR elaborada por ISI, se han aceptado dos artículos en dos revistas internacionales incluidas en la base de datos JCR, se ha enviado un artículo a una revista internacional incluida en la base de datos JCR que está en proceso de revisión, se ha publicado un capítulo de libro en la editorial Springer y también se han publicado dos contribuciones a conferencias internacionales: (i) el tercer Intelligent and Fuzzy Systems Conference, que se publicó como parte de la serie de libros Lecture Notes in Networks and Systems; (ii) el IEEE International Conference on Fuzzy System 2021. En resumen, este informe consta de un total de cinco artículos publicados, aceptados o enviados a prestigiosas revistas internacionales, un capítulo de libro y dos contribuciones a conferencias internacionales.

A continuación se describe brevemente la estructura de esta memoria de investigación:

- **Capítulo 2:** Presenta conceptos preliminares utilizados en nuestra propuesta para alcanzar los objetivos, como la toma de decisiones, la toma de decisiones en grupo, los procesos de alcance de consenso, la toma de decisiones multi-criterio, la toma de decisiones bajo incertidumbre, la toma de decisiones lingüística, los modelos lingüísticos computacionales, los operadores de agregación y las medidas de distancia/similitud simétricas o asimétricas.

- **Capítulo 3:** Presenta brevemente los artículos publicados/aceptados y en revisión, así como un capítulo de libro y dos contribuciones a congresos internacionales, que constituyen la memoria de esta tesis. Para cada publicación, se presenta una breve discusión de los resultados obtenidos.

- **Capítulo 4:** Este capítulo es el núcleo de la tesis doctoral, contiene las publicaciones obtenidas como resultado de esta investigación, y para cada publicación se indica la información sobre los índices de calidad.

- **Capítulo 5:** Expone las conclusiones finales extraídas de este estudio y analiza algunos trabajos futuros como desarrollo del presente estudio.

A.2 Resumen

Como sabemos, la toma de decisiones es un proceso común de los seres humanos para llevar a cabo actividades del mundo real, que suelen definirse en situaciones donde la información es vaga e imprecisa, por lo que existen diferentes enfoques para tratar esta incertidumbre. Uno de los enfoques más utilizados en toma de decisiones es el ELD [90, 91, 92], que proporciona resultados satisfactorios modelando la incertidumbre mediante variables lingüísticas. El uso de información lingüística implica procesos de CP [80, 86, 94, 95]. En la literatura se pueden encontrar distintos modelos lingüísticos computacionales que modelan la información lingüística, sin embargo, la mayoría de ellos utiliza conjuntos de términos lingüísticos definidos a priori que limita a los expertos que participan en los problemas de toma de decisiones a expresar sus opiniones mediante un único término lingüístico. Por esta razón, algunos investigadores han indicado que es necesario proponer nuevos modelos de representación lingüística, que sean capaces de generar expresiones lingüísticas más complejas que los términos lingüísticos simples y que mejoren la flexibilidad para que los expertos expresen sus opiniones. Para evitar estas limitaciones, Rodríguez et al [60, 61] propusieron el uso de gramáticas libre de contexto para generar ELC más flexibles y ricas que los términos lingüísticos simples y cercanas al modelo cognitivos de los seres humanos. Estas expresiones estan basadas en los conjuntos de términos lingüísticos difusos dudosos que modelan la duda de los expertos cuando éstos tienen que expresar sus opiniones y dudan entre varios términos lingüísticos. Junto a este modelo de representación lingüística se han propuesto diferentes modelos computacionales [50, 60]. Sin embargo, la pérdida de información en los procesos de CP y la falta de interpretación de los resultados obtenidos limitan su uso. El modelo lingüístico

2-tupla [35] destaca en estos aspectos por su precisión e interpretabilidad de los resultados debido al uso de la traslación simbólica, sin embargo los valoes lingüísticos 2-tupla siguen estando limitados por el uso de términos lingüísticos simples. Por tanto, la combinación de las ELC y el concepto de traslación simbólica podrían dirigir a un proceso de CP mejorado para ELC. Recientemente, se ha definido un nuevo modelo de representación lingüística para ELC junto con un modelo computacional que manteniene la interpretabilidad y precisión de los resultados. Este modelo extiende las ELC mediante el concepto de traslación simbólica del modelo lingüístico 2-tupla dando lugar a un nuevo modelo de expresiones lingüística comparativa extendida con traslación simbólica (ELICIT) [62]. Estas expresiones extienden la representación de las ELC generadas mediante una gramática libre de contexto a un dominio continuo para realizar los PC sin ningún tipo de aproximación. Para aplicar este modelo en problemas de toma de decisiones, es necesario definir nuevos operadores de agregación, medidas de similitud, modelos de toma de decisión multicriterio, modelos de toma de decisión multicriterio en grupo, modelos de consenso etc. Por tanto, en esta memoria de investigación hemos presentado las siguientes propuestas:

(1) La agregación de expresiones ELICIT es esencial para resolver problemas de toma de decisiones, pero sólo existen dos operadores de agregación definidos para agregar expresiones ELICIT. Por lo tanto, es necesario definir nuevos operadores de agregación para poder modelar una amplia gama de escenarios de decisión. Los operadores de agregación más utilizados en toma de decisiones son entre otros, el operador de agregación de la media aritmética, el operador de agregación de la media ponderada, el operador de la media ponderada ordenada y el operador de agregación de la integral de Choquet. Estos operadores pueden ser extendidos para agregar expresiones ELICIT, lo que naturalmente conduce a la definición de nuevos operadores de agregación así como nuevos modelos de TDMC.

(2) La clasificación de las expresiones ELICIT es una tarea necesaria y compleja, y el reto consiste en calcular la similitud o distancia entre dos expresiones ELICIT. A menudo se calcula mediante la distancia o similitud entre sus envelops difusos, es decir, números difusos trapezoidales. En la literatura existente, se han investigado diferentes métodos para clasificar números difusos, por ejemplo, (i) métodos basados en la defuzzificación; (ii) métodos basados en la distancia entre números difusos o (iii) métodos de comparación por pares. Por tanto, estos métodos pueden aplicarse directamente a la clasificación de las expresiones ELICIT y, a continuación, pueden proponerse varios métodos de clasificación.

(3) Es necesario tener en cuenta que puede haber conflictos entre los expertos del grupo; en consecuencia, los resultados colectivos obtenidos pueden que no sean aceptados por todos los expertos que participan en el problema de toma de decisiones. En otras palabras, algunos expertos consideran que sus opiniones no se han tenido en cuenta. Esto lleva a introducir un PAC antes del ADL. Por otra parte, cabe señalar que las características del problema de decisión definen no sólo la estructura de preferencias, sino también el esquema de solución del análisis de decisión. Las estructuras de preferencias más utilizadas en los problemas de TDG son la ordenación de preferencias, los vectores de utilidad, las relaciones recíprocas de preferencias (RRP), etc. Entre ellas, la RRP es una comparación por pares muy utilizada y su principal ventaja es que los expertos se centran en sólo dos alternativas a la vez, lo que facilita la expresión de sus preferencias. Sin embargo, esta forma de proporcionar preferencias limita la percepción global de las alternativas por parte de los expertos y las preferencias proporcionadas podrían ser inconsistentes. Además, si el número de alternativas a comparar es muy grande, la posibilidad de ser inconsistentes aumenta considerablemente. La inconsistencia en toma de decisiones conduce a resultados irracionales y no fiables. Por tanto, es importante, incluso crítico, estudiar las condiciones en las que una RRP satisfaga la consistencia en problemas de TDG con expresiones ELICIT.

Estas propuestas introducen mejoras de vanguardia para esta memoria de investigación y abordan eficazmente algunos de los retos actuales en toma de decisión.

A.3 Conclusiones y Trabajos Futuros

Por último, esta sección resume la memoria de investigación actual, repasa las principales propuestas y resultados obtenidos a lo largo de esta memoria y sugiere algunos trabajos futuros.

A.3.1 Conclusiones

La toma de decisiones es un proceso humano habitual en muchas actividades del mundo real, como la ingeniería, la medicina, etc. Los problemas de decisión del mundo real suelen definirse en contextos en los que la información es difusa e imprecisa, por lo que existen diferentes enfoques para tratar este tipo de incertidumbre. En esta memoria de investigación, nos centramos en el ELD, que proporciona resultados satisfactorios en problemas de toma de decisiones. El uso del ELD implica procesos de CP. En la literatura, existen diferentes modelos lingüísticos computacionales para modelar la información lingüística, sin embargo, la mayoría de ellos utilizan términos lingüísticos definidos a priori, y limitan a los expertos expresar sus opiniones utilizando un único término lingüístico. Por ello, algunos investigadoes indicaron la necesidad de mejorar la flexibilidad para expresar términos lingüísticos más complejos, ya que a veces el uso de un único término lingüístico no es suficiente para reflejar el conocimiento de los expertos. Para superar estas limitaciones, se introdujo el uso de gramáticas libres de contexto para generar ELC similares al lenguaje natural de los seres humanos capaces de modelar la duda de los expertos. Estas expresiones lingüísticas se basan en el concepto de conjunto de términos lingüísticos difusos dudosos. Sin embargo, la pérdida de información en los procesos de CP y la interpretabilidad de sus resultados limitan el uso de los conjuntos de términos lingüísticos difusos dudosos. Por ello, se introduce un nuevo modelo de representación, denominado expresiones lingüística comparativa extendida con traslación simbólica (ELICIT), junto con un modelo computacional. Este novedoso modelo lingüístico mejora la precisión de los resultados, ya que el proceso de CP mediante traslación simbólica no realiza ninguna aproximación, y es capaz de mantener la interpretabilidad de los resultados. El modelo ELICIT ha sido propuesto recientemente para tratar problemas de decisión, por lo que es necesario proponer nuevos operadores de agregación, medidas de similitud, modelos de TDG, modelos de consenso, etc., que sean capaces de manejar expresiones ELICIT. Los resultados obtenidos son los siguientes:

(1) El proceso de agregación desempeña un papel crucial en el esquema de solución de un problema de toma de decisiones, por lo que en esta memoria de investigación se han propuesto varios operadores de agregación para modelar una amplia gama de escenarios de decisión.

    (a) Utilizando el principio de extensión de Zadeh, el operador OWA de tipo-1 se ha extendido para definir dos operadores de agregación que fusionan expresiones ELICIT, uno de los cuales es el operador ELICIT-OWA de tipo-1. El otro, denominado operador ELICIT-t1-OWA, ha utilizado nuevos pesos intervalares inducidos por nuevos cuantificadores lingüísticos intervalares.

    (b) Del mismo modo, hemos extendido el operador OWA inducido (IOWA) de tipo-1 para fusionar expresiones ELICIT. Se denomina operador ELICIT-t1-IOWA.

    (c) La integral de Choquet se utiliza a menudo para capturar las interrelaciones entre criterios en problemas de toma de decisiones multicriterio, por lo que se han propuesto varios tipos de operadores de integral de Choquet, como el operador Choquet-OWA y el operador Choquet de la media, para fusionar expresiones ELICIT.

(2) Clasificar expresiones ELICIT es una tarea necesaria y compleja, por tanto, el reto es calcular la similitud o distancia entre dos expresiones ELICIT. Teniendo en cuenta que las expresiones ELICIT se manejan normalmente mediante su envelop difuso, es decir, números difusos trapezoidales, hemos ampliado el modelo de relación de Tversky en forma de conjuntos crisp, conjuntos difusos, y vectores, para desarrollar una medida de similitud asimétrica. A continuación, también hemos propuesto una distancia entre números difusos.

(3) También hemos propuesto un modelo de consenso para las expresiones ELICIT, que tiene en cuenta la consistencia de las preferencias de los expertos para obtener resultados fiables y fáciles de entender.

En este sentido, cabe destacar que se han cumplido todos los objetivos identificados al inicio de esta memoria de investigación, proporcionando herramientas, modelos y resultados que mejoran el estado del arte previo a nuestra investigación y ofrecen la posibilidad de seguir investigando, tal y como se describe en los siguientes apartados.

A.3.2 Trabajos Futuros

A partir de los resultados obtenidos en esta investigación, es posible identificar algunas posibles propuestas para continuar con la investigación realizada en esta tesis doctoral. Estos trabajos futuros son los siguientes:

(1) Proponer nuevos métodos para calcular los envelops de las expresiones ELICIT y, a continuación, basándose en estos nuevos métodos, se propondrán nuevos operadores de agregación y medidas de distancia.

(2) Algunos investigadores han señalado que las palabras significan diferentes cosas para diferentes personas, y han introducido varias propuestas para la toma de decisiones mediante el uso de la semántica individual personalizada. Parece muy interesante utilizar la semántica individual personalizada en modelos de consenso que manejen expresiones ELICIT.

(3) A veces hay problemas de toma de decisiones en los que los expertos implicados tienen diferentes conocimientos o experiencia, y es conveniente utilizar modelos lingüísticos multigranulares. Por ello, estos modelos podrían ser extendidos para utilizar expresiones ELICIT.

(4) Definir modelos de toma de decisiones que utilicen información heterogénea, teniendo en cuenta las expresiones ELICIT.

List of Figures

Bibliography

[1] S. Abbasbandy and T. Hajjari. A new approach for ranking of trapezoidal fuzzy numbers. *Computers & Mathematics with Applications*, 57(3):413–419, 2009.

[2] Z. Ali, T. Mahmood, and M.-S. Yang. TOPSIS method based on complex spherical fuzzy sets with bonferroni mean operators. *Mathematics*, 8(10):1739, 2020.

[3] M. Andrea Rodriguez and M. J. Egenhofer. Comparing geospatial entity classes: an asymmetric and context-dependent similarity measure. *International Journal of Geographical Information Science*, 18(3):229–256, 2004.

[4] G. Beliakov, T. Calvo, and S. James. Consensus measures constructed from aggregation functions and fuzzy implications. *Knowledge-Based Systems*, 55:1–8, 2014.

[5] G. Beliakov, A. Pradera, and T. Calvo. *Aggregation Functions: A Guide for Practitioners*. Springer, 2007.

[6] P. P. Bonissone and K. S. Decker. Selecting uncertainty calculi and granularity: An experiment in trading-off precision and complexity. In *Machine intelligence and pattern recognition*, volume 4, pages 217–247. Elsevier, 1986.

[7] G. Bordogna and G. Pasi. A fuzzy linguistic approach generalizing boolean information retrieval: A model and its evaluation. *Journal of the American Society for Information Science*, 44(2):70–82, 1993.

[8] M. Brunelli and M. Fedrizzi. Boundary properties of the inconsistency of pairwise comparisons in group decisions. *European Journal of Operational Research*, 240(3):765–773, 2015.

[9] B. Cao, Q. Yang, J. T. Sun, and Z. Chen. Learning bidirectional asymmetric similarity for collaborative filtering via matrix factorization. *Data mining and knowledge discovery*, 22(3):393–418, 2011.

[10] L. Chen, G. Duan, S. Wang, and J. Ma. A choquet integral based fuzzy logic approach to solve uncertain multi-criteria decision making problem. *Expert Systems with Applications*, 149:113303, 2020.

[11] Z. S. Chen, K. S. Chin, and K. L. Tsui. Constructing the geometric bonferroni mean from the generalized bonferroni mean with several extensions to linguistic 2-tuples for decision-making. *Applied Soft Computing*, 78:595–613, 2019.

[12] F. Chiclana, F. Herrera, and E. Herrera-Viedma. Integrating multiplicative preference relations in a multipurpose decision-making model based on fuzzy preference relations. *Fuzzy sets and systems*, 122(2):277–291, 2001.

[13] F. Chiclana, E. Herrera-Viedma, S. Alonso, and F. Herrera. Cardinal consistency of reciprocal preference relations: a characterization of multiplicative transitivity. *IEEE transactions on fuzzy systems*, 17(1):14–23, 2008.

[14] G. Choquet. Theory of capacities. In *Annales de l'institut Fourier*, volume 5, pages 131–295, 1954.

[15] V. Cutello and J. Montero. Fuzzy rationality measures. *Fuzzy sets and Systems*, 62(1):39–54, 1994.

[16] V. Cutello and J. Montero. Equivalence and compositions of fuzzy rationality measures. *Fuzzy sets and Systems*, 85(1):31–43, 1997.

[17] R. Degani and G. Bortolan. The problem of linguistic approximation in clinical decision making. *International Journal of Approximate Reasoning*, 2(2):143–162, 1988.

[18] M. Delgado, J. L. Verdegay, and M. A. Vila. On aggregation operations of linguistic labels. *International journal of intelligent systems*, 8(3):351–370, 1993.

[19] M. Delgado, M. A. Vila, and W. Voxman. A fuzziness measure for fuzzy numbers: Applications. *Fuzzy sets and systems*, 94(2):205–216, 1998.

[20] M. Delgado, M. A. Vila, and W. Voxman. On a canonical representation of fuzzy numbers. *Fuzzy sets and systems*, 93(1):125–135, 1998.

[21] Y. Dong, Y. Xu, and H. Li. On consistency measures of linguistic preference relations. *European Journal of Operational Research*, 189(2):430–444, 2008.

[22] H. Doukas, C. Karakosta, and J. Psarras. Computing with words to assess the sustainability of renewable energy options. *Expert Systems with Applications*, 37(7):5491–5497, 2010.

[23] D. Dubois. The role of fuzzy sets in decision sciences: Old techniques and new directions. *Fuzzy Sets and Systems*, 184(1):3–28, 2011.

[24] D. Dubois. Fuzzy weighted averages and fuzzy convex sums: Author's response. *Fuzzy Sets and Systems*, 213:106–108, 2013.

[25] B. Dutta, Á. Labella, R. Rodríguez, and L. Martínez. Aggregating interrelated attributes in multi-attribute decision-making with ELICIT information based on Bonferroni mean and its variants. *International Journal of Computational Intelligence Systems*, 12:1179 – 1196, 2019.

[26] P. Eklund, A. Rusinowska, and H. De Swart. Consensus reaching in committees. *European Journal of Operational Research*, 178(1):185–193, 2007.

[27] E. Elboher and M. Werman. Asymmetric correlation: a noise robust similarity measure for template matching. *IEEE Transactions on Image Processing*, 22(8):3062–3073, 2013.

[28] D. Filev and R. R. Yager. On the issue of obtaining owa operator weights. *Fuzzy sets and systems*, 94(2):157–169, 1998.

[29] D. García-Zamora, Á. Labella, R. M. Rodríguez, and L. Martínez. A linguistic metric for consensus reaching processes based on ELICIT comprehensive minimum cost consensus models. *IEEE Transactions on Fuzzy Systems*, pages 1–13, 2022.

[30] M. Grabisch, J. Marichal, R. Mesiar, and E. Pap. *Aggregation Functions*. Cambridge University Press, New York, USA, 2009.

[31] M. Grabisch and M. Roubens. Application of the choquet integral in multi criteria decision making. *Fuzzy Measures and Integrals-Theory and Applications*, pages 348–374, 2000.

[32] F. Herrera, S. Alonso, F. Chiclana, and E. Herrera-Viedma. Computing with words in decision making: foundations, trends and prospects. *Fuzzy optimization and decision making*, 8:337–364, 2009.

[33] F. Herrera and E. Herrera-Viedma. Linguistic decision analysis: steps for solving decision problems under linguistic information. *Fuzzy Sets and systems*, 115(1):67–82, 2000.

[34] F. Herrera, E. Herrera-Viedma, and L. Martínez. A fusion approach for managing multi-granularity linguistic term sets in decision making. *Fuzzy sets and systems*, 114(1):43–58, 2000.

[35] F. Herrera and L. Martínez. A 2-tuple fuzzy linguistic representation model for computing with words. *IEEE Transactions on Fuzzy Systems*, 8(6):746–752, 2000.

[36] E. Herrera-Viedma, F. J. Cabrerizo, J. Kacprzyk, and W. Pedrycz. A review of soft consensus models in a fuzzy environment. *Information Fusion*, 17:4–13, 2014.

[37] E. Herrera-Viedma, F. Herrera, and F. Chiclana. A consensus model for multiperson decision making with different preference structures. *IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans*, 32(3):394–402, 2002.

[38] E. Herrera-Viedma, F. Herrera, F. Chiclana, and M. Luque. Some issues on consistency of fuzzy preference relations. *European journal of operational research*, 154(1):98–109, 2004.

[39] Z. Hua, X. Jing, and L. Martínez. Consensus reaching for social network group decision making with elicit information: A perspective from the complex network. *Information Sciences*, 2023.

[40] B. Irlenbusch, T. Mussweiler, D. J. Saxler, S. Shalvi, and A. Weiss. Similarity increases collaborative cheating. *Journal of Economic Behavior & Organization*, 178:148–173, 2020.

[41] J. KACPRZYK. Group decision-making with a fuzzy majority via linguistic quantifiers. part II: a competitive-like pooling. *Cybernetics and System*, 16(2-3):131–144, 1985.

[42] J. Kacprzyk. Group decision making with a fuzzy linguistic majority. *Fuzzy sets and systems*, 18(2):105–118, 1986.

[43] J. Kacprzyk and M. Fedrizzi. 'soft' consensus measures for monitoring real consensus reaching processes under fuzzy preferences. *Control and Cybernetics*, 15(3-4):309–323, 1986.

[44] J. Kacprzyk and M. Fedrizzi. A 'soft' measure of consensus in the setting of partial (fuzzy) preferences. *European Journal of Operational Research*, 34(3):316–325, 1988.

[45] S. C. Kleene. *Introduction to Metamathematics*. Princeton, NJ, USA: North Holland, 1952.

[46] G. Klir. Fuzzy arithmetic with requisite constraints. *Fuzzy Sets and Systems*, 91(2):165 – 175, 1997.

[47] Á. Labella, R. M. Rodríguez, A. A. Alzahrani, and L. Martínez. A consensus model for extended comparative linguistic expressions with symbolic translation. *Mathematics*, 8(12):2198, 2020.

[48] C.-C. Li, Y. Dong, Y. Xu, F. Chiclana, E. Herrera-Viedma, and F. Herrera. An overview on managing additive consistency of reciprocal preference relations for consistency-driven decision making and fusion: Taxonomy and future directions. *Information Fusion*, 52:143–156, 2019.

[49] J. Li, N. Sang, and C. Gao. Completed local similarity pattern for color image recognition. *Neurocomputing*, 182:111–117, 2016.

[50] H. Liu and R. M. Rodríguez. A fuzzy envelope for hesitant fuzzy linguistic term set and its application to multicriteria decision making. *Information Sciences*, 258:220–238, 2014.

[51] J. Lu and D. Ruan. *Multi-objective group decision making: methods, software and applications with fuzzy set techniques*, volume 6. Imperial College Press, 2007.

[52] L. Martínez and F. Herrera. An overview on the 2-tuple linguistic model for computing with words in decision making: Extensions, applications and challenges. *Information Sciences*, 207:1–18, 2012.

[53] L. Martínez, R. M. Rodriguez, F. Herrera, L. Martínez, R. M. Rodriguez, and F. Herrera. *2-tuple linguistic model*. Springer, 2015.

[54] J. M. Mendel. Computing with words: Zadeh, turing, popper and occam. *IEEE computational intelligence magazine*, 2(4):10–17, 2007.

[55] J. Merigó and M. Casanovas. Using fuzzy numbers in heavy aggregation operators. *International Journal of Information Technology*, 4(3):177–182, 2008.

[56] H. B. Mitchell and D. D. Estrakh. An OWA operator with fuzzy ranks. *International Journal of Intelligent Systems*, 13(1):69–81, 1998.

[57] T. Murofushi and M. Sugeno. An interpretation of fuzzy measures and the choquet integral as an integral with respect to a fuzzy measure. *Fuzzy Sets and Systems*, 29(2):201–227, 1989.

[58] I. Palomares, F. J. Estrella, L. Martínez, and F. Herrera. Consensus under a fuzzy context: Taxonomy, analysis framework afryca and experimental case of study. *Information Fusion*, 20:252–271, 2014.

[59] P. Pirasteh, D. Hwang, and J. Jung. Exploiting matrix factorization to asymmetric user similarities in recommendation systems. *Knowledge-Based Systems*, 83:51–57, 2015.

[60] R. M. Rodriguez, L. Martinez, and F. Herrera. Hesitant fuzzy linguistic term sets for decision making. *IEEE Transactions on fuzzy systems*, 20(1):109–119, 2011.

[61] R. M. Rodríguez, L. Martínez, and F. Herrera. A group decision making model dealing with comparative linguistic expressions based on hesitant fuzzy linguistic term sets. *Information Sciences*, 241:28–42, 2013.

[62] Á. L. Romero, R. M. Rodríguez, and L. Martínez. Computing with comparative linguistic expressions and symbolic translation for decision making: Elicit information. *IEEE Transactions on Fuzzy Systems*, 28(10):2510–2522, 2019.

[63] M. Roubens. Fuzzy sets and decision analysis. *Fuzzy sets and systems*, 90(2):199–206, 1997.

[64] R. Saaty. The analytic hierarchy process—what it is and how it is used. *Mathematical Modelling*, 9(3):161–176, 1987.

[65] S. Saint and J. R. Lawson. *Rules for reaching consensus: a modern approach to decision making*. Pfeiffer, 1994.

[66] S. Santini and R. Jain. Similarity measures. *IEEE Transactions on pattern analysis and machine Intelligence*, 21(9):871–883, 1999.

[67] L. Stefanini, L. Sorini, M. L. Guerra, W. Pedrycz, A. Skowron, and V. Kreinovich. Fuzzy numbers and fuzzy arithmetic. *Handbook of granular computing*, 12:249–284, 2008.

[68] M. Sugeno. Theory of fuzzy integrals and its applications. *Doct. Thesis, Tokyo Institute of technology*, 1974.

[69] M. Sugeno. Fuzzy measures and fuzzy integrals—a survey. In *Readings in fuzzy sets for intelligent systems*, pages 251–257. Elsevier, San Francisco, 1993.

[70] Z. Świtalski. General transitivity conditions for fuzzy reciprocal preference matrices. *Fuzzy Sets and Systems*, 137(1):85–100, 2003.

[71] T. Tanino. Fuzzy preference orderings in group decision making. *Fuzzy sets and systems*, 12(2):117–131, 1984.

[72] T. Tanino. Fuzzy preference relations in group decision making. *Non-conventional preference relations in decision making*, pages 54–71, 1988.

[73] A. Tversky. Features of similarity. *Psychological review*, 84(4):327, 1977.

[74] L. Wang, Y. Wang, and L. Martínez. Fuzzy TODIM method based on alpha-level sets. *Expert Systems with Applications*, 140:112899, 2020.

[75] X. Wang and E. E. Kerre. Reasonable properties for the ordering of fuzzy quantities (I). *Fuzzy sets and systems*, 118(3):375–385, 2001.

[76] Y. Wu, H. Hu, and H. Li. Age-invariant face recognition using coupled similarity reference coding. *Neural Processing Letters*, 50(1):397–411, 2019.

[77] Z. Wu and J. Xu. Managing consistency and consensus in group decision making with hesitant fuzzy linguistic preference relations. *Omega*, 65:28–40, 2016.

[78] R. Yager. On ordered weighted averaging aggregation operators in multi criteria decision making. *IEEE Transactions on Systems, Man, and Cybernetics.*, 18(1):183–190, 1988.

[79] R. Yager. Families of OWA operators. *Fuzzy Sets and Systems*, 59(2):125 – 148, 1993.

[80] R. Yager. Computing with words and information/intelligent systems 2: applications, chapter approximate reasoning as a basis for computing with words, 1999.

[81] R. Yager. Using trapezoids for representing granular objects: Applications to learning and OWA aggregation. *Information Sciences*, 178(2):363 – 380, 2008.

[82] R. Yager. On the fusion of multiple multi-criteria aggregation functions with focus on the fusion of OWA aggregations. *Knowledge-Based Systems*, 191:105216, 2020.

[83] R. R. Yager. On ordered weighted averaging aggregation operators in multi criteria decision making. *IEEE Transactions on systems, Man, and Cybernetics*, 18(1):183–190, 1988.

[84] R. R. Yager. Quantifier guided aggregation using OWA operators. *International Journal of Intelligent Systems*, 11(1):49–73, 1996.

[85] R. R. Yager. Induced aggregation operators. *Fuzzy sets and systems*, 137(1):59–69, 2003.

[86] R. R. Yager. On the retranslation process in zadeh's paradigm of computing with words. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 34(2):1184–1195, 2004.

[87] R. R. Yager and D. P. Filev. Induced ordered weighted averaging operators. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 29(2):141–150, 1999.

[88] L. Zadeh. Fuzzy sets. *Inform Control*, 8:338–353, 1965.

[89] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.

[90] L. A. Zadeh. The concept of a linguistic variable and its application to approximate reasoning (I). *Information Sciences*, 8(3):199 – 249, 1975.

[91] L. A. Zadeh. The concept of a linguistic variable and its application to approximate reasoning (II). *Information Sciences*, 8(4):301 – 357, 1975.

[92] L. A. Zadeh. The concept of a linguistic variable and its application to approximate reasoning (III). *Information Sciences*, 9(1):43 – 80, 1975.

[93] L. A. Zadeh. Fuzzy logic= computing with words. *IEEE transactions on fuzzy systems*, 4(2):103–111, 1996.

[94] L. A. Zadeh. From computing with numbers to computing with words. from manipulation of measurements to manipulation of perceptions. *IEEE Transactions on circuits and systems I: fundamental theory and applications*, 46(1):105–119, 1999.

[95] L. A. Zadeh. *Computing with words in Information/Intelligent systems 1: Foundations*, volume 33. Physica, 2013.

[96] L. A. Zadeh, G. J. Klir, and B. Yuan. *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers*, volume 6. World scientific, 1996.

[97] M. Zarghami, F. Szidarovszky, and R. Ardakanian. A fuzzy-stochastic OWA model for robust multi-criteria decision making. *Fuzzy Optimization and Decision Making*, 7:1 – 15, 2008.

[98] Y. Zhang, C. Yin, Q. Wu, Q. He, and H. Zhu. Location-aware deep collaborative filtering for service recommendation. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 2019.

[99] S. Zhou, F. Chiclana, R. John, and J. Garibaldi. Type-1 OWA operators for aggregating uncertain information with uncertain weights induced by type-2 linguistic quantifiers. *Fuzzy Sets and Systems*, 159(24):3281 – 3296, 2008.

[100] S. Zhou, F. Chiclana, R. John, and J. Garibaldi. Alpha-level aggregation: a practical approach to type-1 OWA operation for aggregating uncertain information with applications to breast cancer treatments. *IEEE Transactions on Knowledge and Data Engineering*, 23(10):1455–1468, 2010.