An Effective Load-Flow Approach Based on Gauss-Newton Formulation

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Abstract **–** Despite that most of power systems can be categorized as well-conditioned, ill-conditioned cases are becoming more frequent. Consequently, developing new robust LF techniques is necessary to efficiently solve these cases. In this paper, an effective Load-Flow (LF) approach based on Gauss-Newton's formulation is proposed. Moreover, efficient strategies for exploring the unsolvable region and calculating the solution space boundary are developed. The proposed LF approach is comprehensively validated using a wide variety of ill-conditioned systems in loadbase conditions, near of the maximum loadability points and considering generator' reactive limits. The studied systems range from 1888-bus to 70000-bus. The results prove the efficiency and superiority of proposed approach over other well-known LF methods.

Keywords: Load-Flow analysis, Gauss-Newton's formulation, ill-conditioned systems, unsolvable region, generators' reactive limits.

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Nomenclature

Acronyms	
LF	Load-Flow
NR	Newton-Raphson
ODE	autonomous differential equations
GN	Gauss-Newton
MLP	Maximum loadability point
GN-LF	Gauss-Newton's Load-Flow method
ROA	Region of attraction
Indices and dimensions	
i, j	bus indices
n_g	number of PV buses
n_c	number of PQ buses
n	size of the LF unknown vector ($n = n_g + 2n_c$)
n_{bus}	number of buses ($n_{bus} = n_g + n_c + 1$)
k	iteration number
Variables	
P_i	injected active power at i^{th} bus
Q_i	injected reactive power at i^{th} bus
P_i^{sch}	scheduled active power at i^{th} bus
Q_i^{sch}	scheduled reactive power at i^{th} bus
$V_i\angle\delta_i$	voltage phasor at i^{th} bus
$Y_{ij}\angle\theta_{ij}$	ij^{th} element of admittance matrix
λ	damping factor of the regularized GN method, $\lambda \in \mathbb{R}^+$
μ	step size, $\mu \in \mathbb{R}^+$
ϕ_i	power factor at i^{th} bus
Matrix and vectors	
$\boldsymbol{R}$	second order term
$\boldsymbol{A}$	approximation of Hessian matrix
$\boldsymbol{I}$	identity matrix
$\boldsymbol{B}$	coefficients matrix of the augmented LF problem for calculating the solution space boundary
$\boldsymbol{S}_{sch}$	submatrix of $\boldsymbol{B}$
$\boldsymbol{\Gamma}$	submatrix of $\boldsymbol{B}$
$\boldsymbol{x}$	LF unknown vector, $\boldsymbol{x} = \left[\delta_1, \dots, \delta_{n_c}, \dots, \delta_{n_g} \middle| V_1, \dots, V_{n_c}\right]^T$
$\boldsymbol{s}$	vector of scheduled power, $\boldsymbol{s} = \left[P_1^{sch}, \dots, P_{n_c}^{sch}, \dots, P_{n_g}^{sch} \middle| Q_1^{sch}, \dots, Q_{n_c}^{sch}\right]^T$
$\boldsymbol{\varepsilon}$	error vector of the standard LF problem
$\boldsymbol{x^m}$	local minimum of SSR
$\boldsymbol{v}$	right eigenvector corresponding to a zero eigenvalue of the singular LF Jacobian matrix
$\boldsymbol{z}$	unknown of the augmented LF problem vector for calculating the solution space boundary, $\boldsymbol{z} = [\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{\rho}]^T$
$\boldsymbol{\rho}$	vector of loading factors, $\boldsymbol{\rho} = \left[\rho_{P_1}, \dots, \rho_{P_{n_{bus}-1}}\right]^T$
$\boldsymbol{s_\rho}$	vector of scheduled power for the loading factor ρ, $\boldsymbol{s_\rho} = \left[\rho_{P_1} P_1^{sch}, \dots, \rho_{P_{n_c}} P_{n_c}^{sch}, \dots, \rho_{P_{n_g}} P_{n_g}^{sch} \middle| \tan\phi_1\, \rho_{P_1} Q_1^{sch}, \dots, \tan\phi_{n_c}\, \rho_{P_{n_c}} Q_{n_c}^{sch}\right]^T$
$\mathbf{f}$	LF equations, $\boldsymbol{f} = \left[P_1, \dots, P_{n_c}, \dots, P_{n_g} \middle| Q_1, \dots, Q_{n_c}\right]^T$
$\boldsymbol{W}$	sum of squares of residuals (SSR) of the standard LF problem
$\boldsymbol{V}$	sum of squares of residuals (SSR) of the augmented LF problem for calculating the solution space boundary
$\boldsymbol{E}$	error vector of the augmented LF problem for calculating the solution space boundary, $\boldsymbol{E}(\boldsymbol{z}) = [\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{s_\rho}, \nabla_x^T \boldsymbol{f}(\boldsymbol{x})\boldsymbol{v}, \boldsymbol{v}^T\boldsymbol{v} - 1]^T$
Sets	
$\mathbb{Q}^+$	generators' indices whose higher reactive limit violated
$\mathbb{Q}^-$	generators' indices whose lower reactive limit violated
Operators	
$\nabla_{\boldsymbol{x}}$	Jacobian matrix with respect $\boldsymbol{x}$
$\nabla_{\boldsymbol{xx}}^2$	Hessian matrix with respect $\boldsymbol{x}$ two times
$(\cdot)^*$	adjoint

$(\cdot)^{-1}$	inverse
$(\cdot)^{T}$	transpose
Constants	
α	$\alpha \in [0, {}^{1}/_{2})$
β	$\beta \in [{}^{1}/_{2}, 1)$
sub	if it is equal to 1, the subroutine for avoiding the divergence beyond the MLP is activated
tol	convergence tolerance, $tol \in \mathbb{R}^{+}$

1. Introduction

The LF analysis is widely used in planning and operation of power systems. This tool aims to determine the steady state of a power system at specified set of loads and generators. Frequently, the solution of LF problem is achieved by employing some nonlinear techniques. The most popular LF methods are NR and Fast-Decoupled techniques [1]. When LF equations are ill-conditioned, conventional techniques do not offer good performance. This is due to numerical instability of the method rather than LF equations. This fact provokes that standard solution methods become unstable, and they may diverge. It means that algorithm evolves further away to the solution, instead of approximating it. Consequently, solution is not achieved. In some papers, ill-conditioned systems are identified through its condition number, so we can claim that a system is ill-conditioned if its condition number is properly high. Therefore, an appropriate definition for ill-conditioned power systems (and here preferred) has been provided in [2]. Indeed, a power system can be categorized as ill-conditioned if, despite that solution of the LF problem exists, it is not reachable using standard methods and a flat start.

Robust techniques aim to improve the numerical stability of standard methods. Although most of cases are well-conditioned, ill-conditioned systems are becoming more frequent [3]. Therefore, developing new robust techniques for solving the LF problem for ill-conditioned systems is always justified and well-received.

In [4], a robust methodology was proposed. In this methodology, an optimal multiplier is calculated during the iterative process which aims to modify the incremental vector in order to avoid the divergence. In [2], the Continuous Newton's method has been extended to solve the

LF problem, where the formal analogy between it and a set of ODEs has been established. On the basis of this analogy, any numerical technique can be applied to find the solution of LF problem, this issue has been also recently exploited in [5]. A dynamic solution paradigm has been proposed in [3]. It is based on raising the LF problem as a set of ODEs. Thus, the LF problem can be solved using integration techniques. The Levenberg methods have been recently applied to solve the LF problem. Results reported in [6, 7], show that Levenberg-based LF solvers may be a good alternative for addressing the ill-conditioned systems. However, some drawbacks have been mentioned in [8], for instance, standard Levenberg-Marquardt's method often employs too many iterations to converge. Recently, several LF methods based on numerical techniques have been developed. In [9], three robust LF techniques based on the Adams-Bashforth's methods has been proposed. On the other hand, the Bulirsch-Stoer method has been exploited for solving the LF problem in [10]. Finally, the solution of ill-conditioned systems has been addressed by developing a combined approach in [11].

The main target of this paper is to solve the ill-conditioned power systems which most of existing robust LF techniques may be not efficient enough to properly manage this kind of systems. To do that, the alternative raising of the LF as an unconstrained minimization problem is considered. It is worth to mention that transforming the LF problem in an equivalent unconstrained minimization problem is not a novel idea. Nevertheless, authors have detected a lack of deeper research efforts in this field. In fact, to the best of our knowledge, only the Levenberg's method has only been exploited as minimization technique for solving the LF. In order to fill this gap. In this paper, the GN (other well-known minimization method) is studied for LF analysis. Motivation on the usage of the GN method relapses in its outstanding robustness besides its efficiency. Enhancing features of proposed approach have been also shown, such as its ability to manage equipment's reactive limits and calculate a valuable solution of the problem beyond MLP. In addition, an extended LF problem for calculating the solution space boundary has been proposed. Based on the obtained results, the authors believe that the proposed GN-LF approach supposes a considerable contribution for obtaining the LF solution in ill-conditioned systems. To sum up, the main contributions of this paper are summarized in the following points:

- A robust and an efficient LF approach based on GN formulation is proposed;
- A subroutine for avoiding the divergence of proposed approach beyond the MLP is developed. It allows to obtain an approximate solution of the LF problem, which can be used for different purposes;
- An augmented LF problem is proposed. This formulation allows to directly calculate the MLP and the solution space boundary. The proposed methodology offers good performance in this problem, which is often ill-conditioned;
- The proposed approach is validated in a huge variety of ill-conditioned systems, ranging from 1888- to 70000-buses, under different scenarios;
- The performance of proposed approach is compared with other 9 standard and robust LF techniques proposed in the literature.

Remainder of this paper is organized as follows: The LF problem and its solution paradigm as an unconstrained minimization problem are explained in Section 2. Section 3 describes the solution of LF problem using the proposed GN-LF approach, moreover its potential application and enhancing features are discussed. Several numerical experiments are analysed in Section 4. Finally, the main conclusions are mentioned in Section 5.

2. Theoretical background

2.1.- *Brief description of LF problem*

Formally, the LF problem in polar coordinates is a set of nonlinear equations which relates the injected power nodal with the nodal voltages as follows:

$$P_i = \sum_{j=1}^{n_g+n_c} |V_i| |V_j| |Y_{ij}| \cos(\theta_{ij} - \delta_i + \delta_j) \quad (1)$$

$$Q_i = \sum_{j=1}^{n_c} |V_i| |V_j| |Y_{ij}| \sin(\theta_{ij} - \delta_i + \delta_j) \quad (2)$$

Therefore, the solution of LF problem consists of solving a set of n nonlinear equations which can be stated as follows:

$$\mathbf{f}(\boldsymbol{x}) = \boldsymbol{s} \quad (3)$$

Since (3) are nonlinear equations, an iterative technique must be used for solving them. Thus, at k^{th} iteration, an error vector is defined as follows:

$$\boldsymbol{\varepsilon}^{(k)} = \mathbf{f}(\boldsymbol{x}^{(k)}) - \boldsymbol{s} \quad (4)$$

2.2.- Solution of LF problem as an unconstrained minimization problem

If the solution of (3) exists, zeroing (4) is equivalent to minimize the sum of squares of residuals as follows:

$$\min_{\boldsymbol{x}} \boldsymbol{W}(\boldsymbol{x}) = \frac{1}{2} \boldsymbol{\varepsilon}^T \boldsymbol{\varepsilon} \quad (5)$$

A local minimum of (5) is achieved at a point $\boldsymbol{x^m}$, where the gradient is zero, equivalently:

$$\nabla_{\boldsymbol{x}} \boldsymbol{W}(\boldsymbol{x^m}) = \boldsymbol{\varepsilon}^T \nabla_{\boldsymbol{x}} \mathbf{f}(\boldsymbol{x^m}) = 0 \quad (6)$$

In fact, two situations may occur [12]:

- $\boldsymbol{\epsilon} = 0$ and consequently $\mathbf{f}(\boldsymbol{x^m}) = \boldsymbol{s}$, it means that $\boldsymbol{x^m}$ is the solution of LF problem (3).
- $\boldsymbol{\epsilon} \neq 0$ but $\nabla_{\boldsymbol{x}} \boldsymbol{W}(\boldsymbol{x^m}) = 0$, it occurs when $\boldsymbol{x^m}$ is the local minimum of (10) but not a solution of (3). In this case, the LF Jacobian matrix $\nabla_{\boldsymbol{x}} \mathbf{f}(\boldsymbol{x^m})$ is singular at this point.

The second situation is intriguing since $\boldsymbol{x^m}$ lacks of physical meaningful. This situation occurs when the loading level is higher than the MLP. In this case, $\boldsymbol{x^m}$ corresponds to the best possible solution of (3) at this loading level. Therefore, one can deduce that when the LF is solved beyond the MLP, the solution of (5) is driven to find that point where the Jacobian matrix becomes singular.

3. Proposed GN-LF approach

Finding the stationary points of (5), is equivalent to solve the following nonlinear gradient equation [13]:

$$\nabla_x \boldsymbol{W}(\boldsymbol{x}) \coloneqq \nabla_x \mathbf{f}(\boldsymbol{x}) \nabla_x^* \mathbf{f}(\boldsymbol{x}) = 0 \tag{7}$$

Rightly, Newton's method can be applied for solving (7) [14]. In a generic way, the k^{th} iteration of Newton's method for solving (7) is defined as follows:

$$\nabla_{xx}^2(\boldsymbol{x}^{(k)}) \coloneqq \nabla_x \mathbf{f}(\boldsymbol{x}^{(k)}) \nabla_x^* \mathbf{f}(\boldsymbol{x}^{(k)}) + \boldsymbol{R}^{(k)} \tag{8}$$

Newton's method is computationally inefficient due to it needs to update the Hessian matrix and the second order term each iteration [13]. Alternatively, GN employs a positive approximation of $\nabla_{xx}^2 \mathbf{f}(\boldsymbol{x})$ [15]. Applicability of this technique to solve the LF problem is straightforward if the generic k^{th} iteration of proposed GN-LF is defined as follows [16]:

$$\Delta \boldsymbol{x}^{(k)} = -\left[\boldsymbol{A}^{(k)}\right]^{-1} \nabla_x^* \mathbf{f}(\boldsymbol{x}^{(k)}) \boldsymbol{\varepsilon}^{(k)} \tag{9}$$

There are several ways to calculate $\boldsymbol{A}$. This matrix can be the Moore-Penrose pseudoinverse of the Hessian matrix [17]. On the other hand, it can be calculated as $\boldsymbol{A} = \nabla_x^* \mathbf{f}(\boldsymbol{x}^{(k)}) \nabla_x \mathbf{f}(\boldsymbol{x}^{(k)})$ or $\boldsymbol{A} = \nabla_x^* \mathbf{f}(\boldsymbol{x}^{(0)}) \nabla_x \mathbf{f}(\boldsymbol{x}^{(0)})$ for the so-called Inexact GN method [18]. In this paper, the second option is preferred due to the following reasons:

- The main advantage of using $\boldsymbol{A} = \nabla_x^* \boldsymbol{f}(\boldsymbol{x}^{(0)}) \nabla_x \boldsymbol{f}(\boldsymbol{x}^{(0)})$ is avoiding the factorization of Jacobian matrix each iteration. However, if a regularization scheme is introduced, this advantage will be lost.
- The use of $\boldsymbol{A} = \nabla_x^* \mathbf{f}(\boldsymbol{x}^{(k)}) \nabla_x \mathbf{f}(\boldsymbol{x}^{(k)})$ always provides a better incremental vector since the most updating Jacobian matrix is used. Thus, the convergence is normally speed up.

Anyway, the calculation of Moore-Penrose pseudoinverse is heavier than remainder options.

Therefore, equation (9) can be particularized as follows:

$$\Delta \boldsymbol{x}^{(k)} = -\left[\nabla_x^* \mathbf{f}(\boldsymbol{x}^{(k)}) \nabla_x \mathbf{f}(\boldsymbol{x}^{(k)})\right]^{-1} \nabla_x^* \mathbf{f}(\boldsymbol{x}^{(k)}) \boldsymbol{\varepsilon}^{(k)} \quad (10)$$

If the operator $\nabla_x^* \boldsymbol{f}(\boldsymbol{x}) \nabla_x \boldsymbol{f}(\boldsymbol{x})$ is not boundedly invertible, it is necessary to introduce a regularization scheme. To address this issue, a sequence of positive numbers $\lambda^{(k)}, \lambda^{(k)} \mapsto 0$ [19] is introduced and the iterative procedure (10) is rewritten as follows:

$$\Delta \boldsymbol{x}^{(k)} = -\left[\nabla_x^* \mathbf{f}(\boldsymbol{x}^{(k)}) \nabla_x \mathbf{f}(\boldsymbol{x}^{(k)}) + \lambda^{(k)} \boldsymbol{I}\right]^{-1} \left(\nabla_x^* \mathbf{f}\left(\boldsymbol{x}^{(k)}\right) \boldsymbol{\varepsilon}^{(k)} + \lambda^{(k)} (\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(0)})\right) \quad (11)$$

As mentioned before, the advantages of the use $\boldsymbol{A} = \nabla_x^* \boldsymbol{f}(\boldsymbol{x}^{(0)}) \nabla_x \boldsymbol{f}(\boldsymbol{x}^{(0)})$ are lost when a regularization scheme is introduced. It can be easily noted in (11), since the value of λ is updated each iteration, factorization is not avoided.

The value of λ should be taken initially large in order to reduce the impact of $\nabla_x^* \mathbf{f}(\boldsymbol{x}) \nabla_x \mathbf{f}(\boldsymbol{x})$, thus, the eigenvalues of $\nabla_x^* \mathbf{f}(\boldsymbol{x}^{(k)}) \nabla_x \mathbf{f}(\boldsymbol{x}^{(k)}) + \lambda^{(k)} \boldsymbol{I}$ are initially larger and equal to λ. Progressively, it should be reduced until almost zero, hence, the convergence is accelerated and tends to be quadratic. The updating value of λ each iteration can be achieved as follows:

$$\lambda^{(k+1)} = \lambda^{(k)} e^{-2k} \quad (12)$$

3.1.- *Convergence analysis of proposed GN-LF approach*

The convergence properties of GN has been widely studied in the literature from the local [20], or semi-local point of view [21]. Generally, one can affirm that the GN-LF converges if the Jacobian matrix satisfies the Lipschitz condition and the initial guess is sufficiently near to the solution [22].

In order to assess the convergence properties of proposed GN-LF approach, the ROA associated with this approach is calculated and compared with that obtained by the standard NR. Calculation of the ROA of LF problem is a hard task since it cannot be calculated analytically [23], where many LF problems must be solved for different initial guesses.

The ROAs of proposed approach and standard NR have been calculated for the illustrative 2-bus case shown in Fig. 1 [3], taking the initial voltage angle and magnitude of bus #2 (PQ bus) as degrees of freedom. Fig. 2 shows the calculated ROAs for the standard NR and proposed GN-LF, respectively (white surfaces). It is easily noticeable that the proposed GN-LF is more robust than NR since its ROA is wider. It means that the proposed GN-LF is barely affected by the initial guess considered.

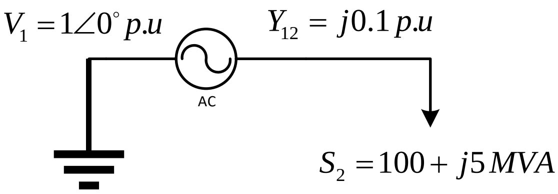

Fig 1 2-bus illustrative system [3]

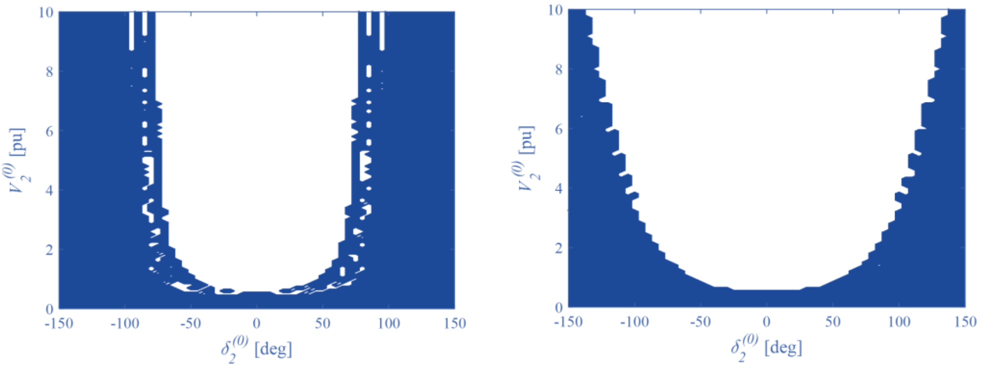

Fig 2 ROAs in the illustrative 2-bus example using NR (left) and proposed GN-LF (right)

Regarding to the order of convergence, it is strongly affected by the value of λ and the strategy adopted for its updating. However, the iterative procedure (11) becomes the standard NR at $\lambda = 0$, therefore, it can be claimed that the GN-LF has a maximum order of convergence of 2, since any $\lambda > 0$ implies smaller order of convergence.

3.2.- *Considering generator's reactive limits*

Handling the generator's reactive limits is an important issue in LF calculation. Basically, the problem consists of determining those solutions subject to all reactive limits.

The proposed GN-LF approach allows to directly incorporate any technique which is currently used in the standard NR. For example, the following typical technique:

<u>Step 1:</u> Solve the LF problem using the proposed GN-LF and a flat start. If there is no PV buses and LF has not found a solution, the problem is unfeasible.

<u>Step 2:</u> Check if any reactive limit is violated.

<u>Step 3:</u> Create a $\mathbb{Q}^{+}$ set with these generators' indices whose higher limit is violated.

<u>Step 4:</u> Create a $\mathbb{Q}^{-}$ set with these generators' indices whose lower limit is violated.

<u>Step 5:</u> Convert those generators $\in [\mathbb{Q}^{+}, \mathbb{Q}^{-}]$ to PQ buses.

<u>Step 6:</u> For each $j \in \mathbb{Q}^{+}$ set $Q_j = Q_j^{max}$

<u>Step 7:</u> For each $j \in \mathbb{Q}^{-}$ set $Q_j = Q_j^{min}$. Return to Step 1.

3.3.- *Exploring the unsolvable space*

Stationary points of (5) are not solutions of the LF problem. Nevertheless, it may be interesting calculating them. For example, in [12, 24], these points allow us to determine the best direction for modifying the load-generation profile to return to the solvable region.

Points of the unsolvable region are in fact solutions of the problem (5) where the LF Jacobian matrix singular. Calculating these points may be problematic for standard techniques since they tend to diverge due to the singularity of Jacobian matrix, thus, solution of (5) would not be achieved. In order to avoid it, we use the so-called Armijo-Goldstein conditions:

$$\boldsymbol{W}(\boldsymbol{x} + \Delta\boldsymbol{x}) > \boldsymbol{W}(\boldsymbol{x}) + \mu\alpha\nabla^T\boldsymbol{W}(\boldsymbol{x})\Delta\boldsymbol{x} \tag{13}$$

$$\boldsymbol{W}(\boldsymbol{x} + \Delta\boldsymbol{x}) < \boldsymbol{W}(\boldsymbol{x}) + \mu\beta\nabla^T\boldsymbol{W}(\boldsymbol{x})\Delta\boldsymbol{x} \tag{14}$$

Equations (13) and (14) allow us to guess when the algorithm is evolving further away to the solution and, consequently, it will diverge. Rightly, we can use (13) and (14) for avoiding the divergence of the algorithm and, therefore, determining the approximate solution of the LF problem beyond of the MLP. In this case, when (13) or (14) are not satisfied, (11) is properly truncated (in a similar way that Iwamoto's method [4]). It is worth to mention that, making use of the Armijo-Goldstein conditions, other alternative sub-routines can be developed. Authors are currently working toward further investigate this point.

By making use of (13) and (14), we propose the following sub-routine for avoiding the divergence of proposed GN-LF in the unsolvable region:

Step 1: Calculate $\Delta\boldsymbol{x}^{(k)}$ using (11).

Step 2: If $sub = 1$, then set $\mu = 1$

Step 3: If (13) and (14) are satisfied then go to step 5. Otherwise go to Step 4.

Step 4: Set $\mu = 0.9\mu$ and $\Delta\boldsymbol{x}^{(k)} = \mu\Delta\boldsymbol{x}^{(k)}$. Return to Step 3.

Step 5: Continue with the LF standard procedure taking $\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \Delta\boldsymbol{x}^{(k)}$

Authors think that calculating these points in an efficient way is interesting for different LF analysis. For instance, the distance of the actual operating point of the MLP may be determined by using the proposed subroutine as it is demonstrated in Section 4.5.

Consequently, the solution process of load flow problem using the proposed GN-LF approach is summarized in the flowchart of Fig. 3.

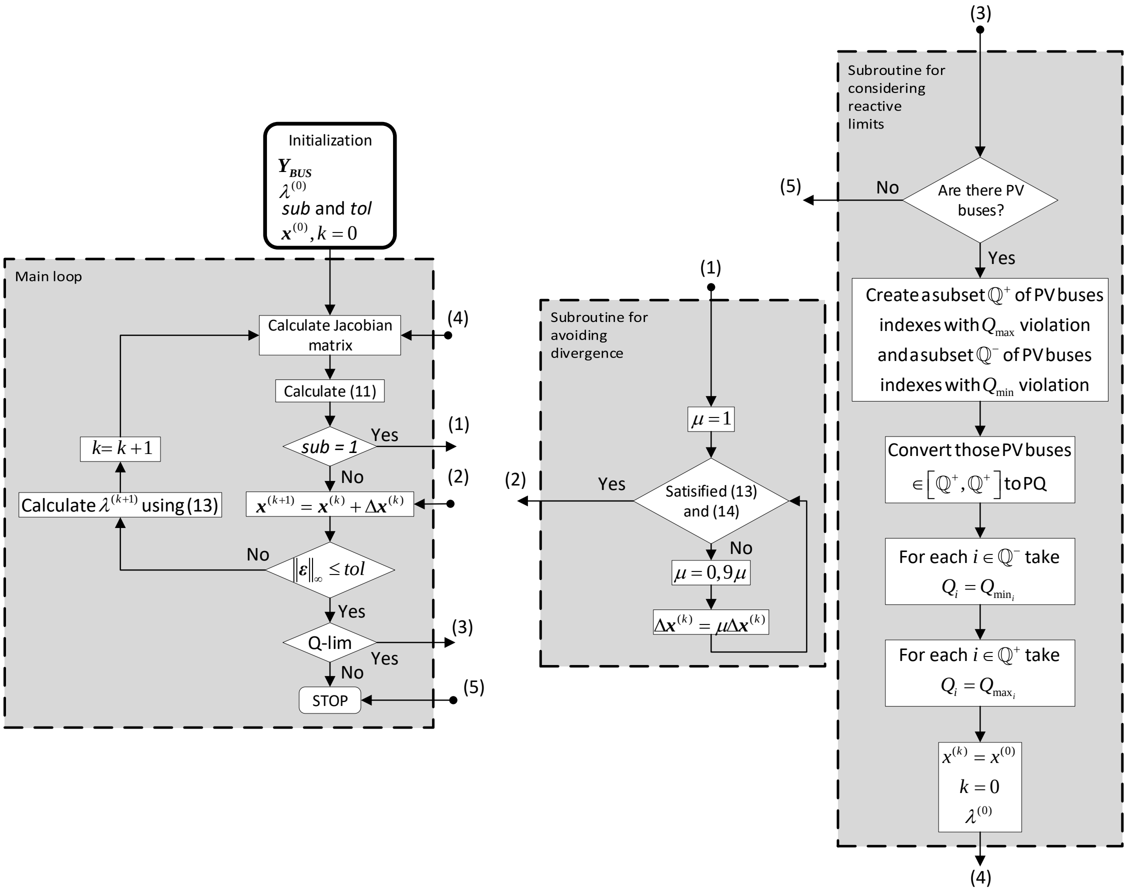

Fig 3 Flowchart of proposed GN-LF approach

3.4.- *Determining the solution space boundary*

Set of points on the boundary of the solvable and unsolvable regions are in fact solutions of the LF problem. However, these set of points must also satisfy the following conditions [25]:

$$\nabla_x^T \mathbf{f}(\boldsymbol{x})\boldsymbol{v} = \mathbf{0} \tag{15}$$

$$\boldsymbol{v}^T \boldsymbol{v} = 1 \tag{16}$$

Therefore, points on the boundary are minimums of the following function:

$$\boldsymbol{V}(\boldsymbol{z}) = \frac{1}{2}\boldsymbol{E}^T(\boldsymbol{z})\boldsymbol{E}(\boldsymbol{z}) \tag{17}$$

where:

$$\boldsymbol{z} = [\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{s}]^T \tag{18}$$

$$\boldsymbol{E}(\boldsymbol{z}) = [\boldsymbol{\varepsilon}, \nabla_x^T \mathbf{f}(\boldsymbol{x})\boldsymbol{v}, \boldsymbol{v}^T\boldsymbol{v} - 1]^{\boldsymbol{T}} \tag{19}$$

In order to simplify the problem, we consider the power factor is kept constant in all load buses, hence we have:

$$\left.\begin{aligned} P_i^{sch} &= \rho_{P_i} P_i^{sch} \\ Q_i^{sch} &= \tan\phi_i \, \rho_{P_i} Q_i^{sch} \end{aligned}\right\} \text{ for load buses} \tag{20}$$

$$P_i^{sch} = \rho_{P_i} P_i^{sch} \text{ for generator buses} \tag{21}$$

Thus, the vector of load factors can be defined as follows:

$$\boldsymbol{\rho} = \left[\rho_{P_1}, \dots, \rho_{P_{n_{bus-1}}}\right]^T \tag{22}$$

The unknown vector (18) and error function (19) become as:

$$\boldsymbol{z} = [\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{\rho}]^T \tag{23}$$

$$\boldsymbol{E}(\boldsymbol{z}) = [\mathbf{f}(\boldsymbol{x}) - \boldsymbol{s}_{\boldsymbol{\rho}}, \nabla_x^T \mathbf{f}(\boldsymbol{x})\boldsymbol{v}, \boldsymbol{v}^T \boldsymbol{v} - 1]^{\boldsymbol{T}} \tag{24}$$

Therefore, the point on the solvable space boundary, is the iterative solution of following nonlinear system:

$$\boldsymbol{B}(\boldsymbol{z})\boldsymbol{z} = -\boldsymbol{E}(\boldsymbol{z}) \tag{25}$$

where:

$$\boldsymbol{B}(\boldsymbol{z}) = \begin{bmatrix} \nabla_x \mathbf{f}(\boldsymbol{x}) & \mathbf{0} & \boldsymbol{S}^{sch} \\ \nabla_x(\nabla_x \mathbf{f}(\boldsymbol{x})\boldsymbol{v}) & \nabla_x \mathbf{f}(\boldsymbol{x}) & \mathbf{0} \\ \mathbf{0} & 2\boldsymbol{v} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Gamma} \end{bmatrix} \in \mathbb{R}^{(2n+n_{bus})\times(2n+n_{bus})} \tag{26}$$

$$\boldsymbol{S}^{sch} = \begin{bmatrix} P_i^{sch} & & \\ & \ddots & \\ & & P_n^{sch} \\ \tan\phi_i \, P_i^{sch} & & \\ & \ddots & \\ & & \tan\phi_{n_c} \, P_{n_c}^{sch} \end{bmatrix} \tag{27}$$

$$\boldsymbol{\Gamma} = \begin{bmatrix} \tan\phi_i & 1 \\ \vdots & \vdots \\ \tan\phi_{n_c} & 1 \end{bmatrix} \tag{28}$$

Consequently, a generic k^{th} iteration of GN-LF for solving the proposed system (25) can be given as:

$$\Delta \boldsymbol{z}^{(k)} = -\left[\boldsymbol{B}^{*}(\boldsymbol{z}^{(k)})\boldsymbol{B}(\boldsymbol{z}^{(k)}) + \lambda^{(k)}\boldsymbol{I}\right]^{-1}\left(\boldsymbol{B}^{*}(\boldsymbol{z}^{(k)})\boldsymbol{E}\left(\boldsymbol{z}^{(k)}\right) + \lambda^{(k)}(\boldsymbol{z}^{(k)} - \boldsymbol{z}^{(0)})\right) \quad (29)$$

The main computational burden of proposed procedure is updating and factorizing of matrix (26). It can be easily checked that system (25) may be almost three times bigger than the standard LF problem (3) and the LF hessian matrix needs to be calculated each iteration, nevertheless, matrix (26) has normally a very sparse structure

Moreover, matrix (26) frequently tends to be ill-conditioned. Nevertheless, the GN-LF is robust enough to easily manage system (25) overcoming potential issues of other methodologies (e.g. NR).

It is worth to notice that the solution of system (25) for a determining vector $\boldsymbol{\phi}$, corresponds with the MLP of the system, which allows to calculate this point directly, oppositely to other techniques.

4. Numerical experiments

In this section, the proposed GN-LF approach is validated using several test systems ranged from 1888-bus to 70000-bus and compared with other well-known LF methods. Different scenarios (ill-conditioned cases, systems operate near of the MLP, considering generator' reactive limits, and measuring the degree of unsolvability) are considered as follows:

4.1.- *Performance of proposed GN-LF approach (ill-conditioned systems)*

One of the most outstanding features of proposed GN-LF approach is its robustness against the other LF methods. In order to demonstrate this robustness, several ill-conditioned systems from the RTE French [26], EU Pegase project [26, 27], the transmission Polish system snapshots [28] and several synthetic grid models from the Texas A&M University [29] have been considered.

The proposed GN-LF approach and other studied techniques have been coded using M-file and implemented in MATPOWER 6.0 [30]. All cases have been solved using a flat initial guess. All simulations have been run under Windows 10 on a 3.4 GHz Intel Core i5-7500 CPU personal computer.

Fig. 4 shows the total number of iterations and execution time of proposed GN-LF for all studied cases under different values of initial value λ and convergence tolerances. As can be seen, the total number of iterations is increased when the initial value of λ is high. The number of iterations does not vary considerably among cases. The execution time is increased with the size of system, ranging from 0.09 s to 6.5 s.

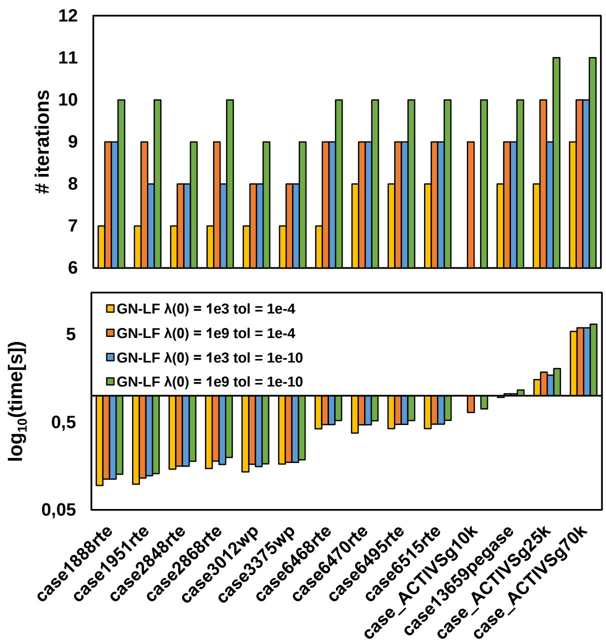

Fig 4 Overall performance of proposed GN-LF approach for solving the LF problem of the studied cases

4.2.- *Performance of proposed GN-LF approach (systems operate near of the MLP)*

In this subsection, the proposed GN-LF approach is tested when the system operates near of the MLP. Fig. 5 shows the required total number of iterations and execution time of proposed GN-LF approach for solving the studied systems when they operate near from their MLP. Similar conclusions to those extracted from Fig. 4 can be extrapolated here.

Nevertheless, the total number of iterations is generally increased due to the condition of the system is worse.

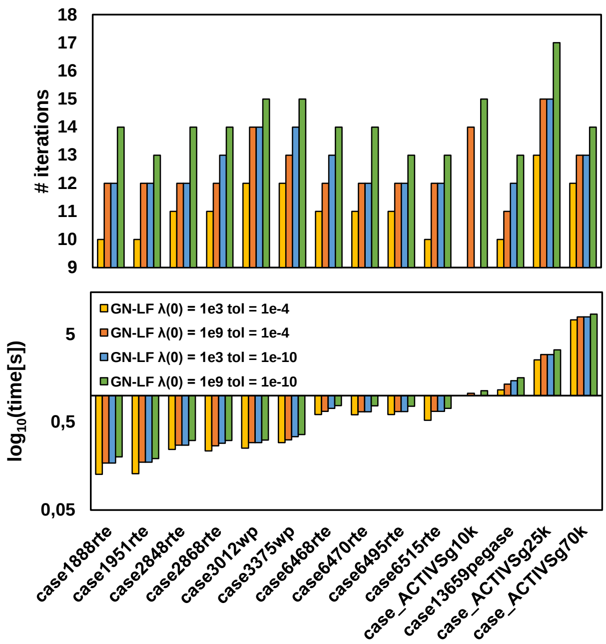

Fig 5 Overall performance of proposed GN-LF for solving the LF problem of the studied cases near of the MLP

4.3.- *Performance of proposed GN-LF approach (considering generators' reactive limits)*

In this subsection, the proposed GN-LF approach is tested considering the generators' reactive limits and the obtained results are shown in Fig. 6. In this test, the overall performance mainly depends on the total number of LF solutions required to find a feasible solution.

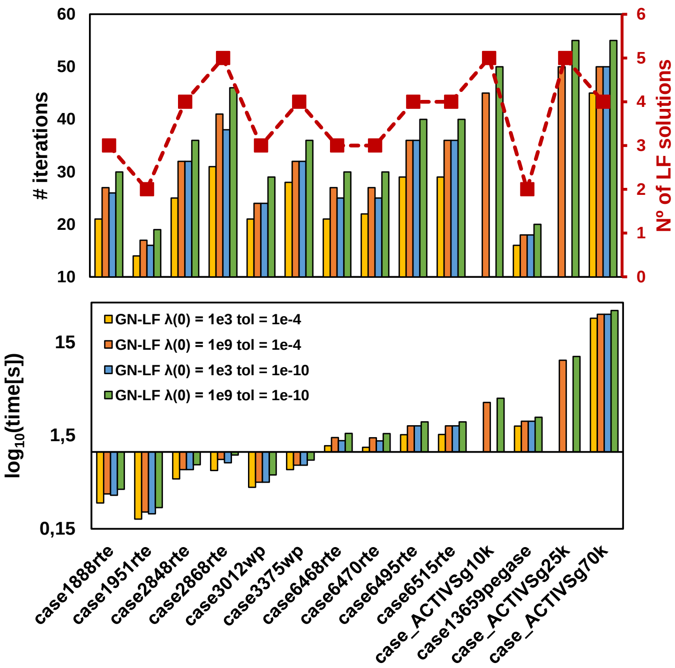

Fig 6 Overall performance of proposed GN-LF for solving the LF problem of the studied cases considering reactive limits

4.4.- *Performance of proposed GN-LF approach (combining heavy loading conditions with generators' reactive limits enforcement)*

In this subsection, an extreme situation is considered. Indeed, we suppose that generators' reactive power limits are enforced, and the system is operated near of the MLP. Results for this scenario are shown in Fig. 7.

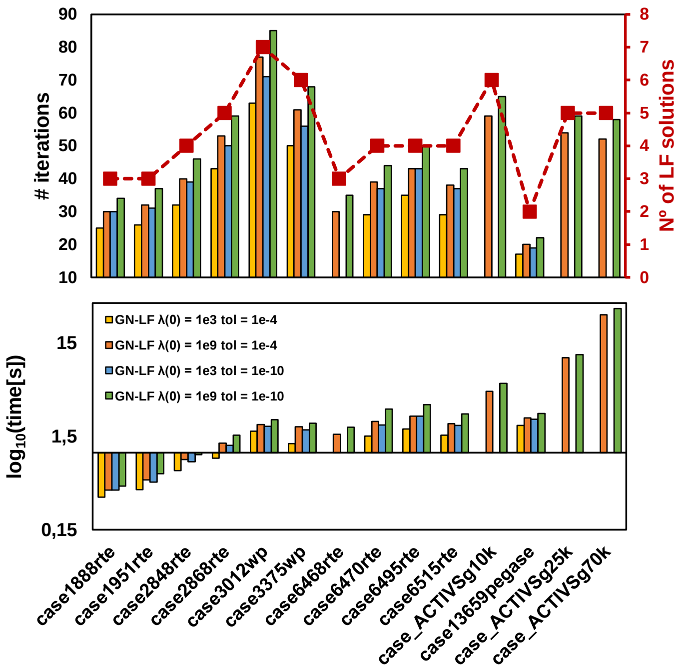

Fig 7 Overall performance of proposed GN-LF for solving the LF problem of the studied cases considering reactive limits and heavy loading conditions

4.5.- *Performance of proposed GN-LF approach compared with other LF techniques*

With the aim to compare the proposed GN-LF with other well-known LF techniques, several robust and standard techniques and considered. The standard NR and other techniques [2, 4, 6, 7, 9, 10] are used for solving the Polish system snapshots (i.e. the 3012wp and 3375wp cases [28]) and the 13659-bus potion of the European system (i.e. the 13659pegase case [26, 27]). The required number of iterations and computation times of these techniques are reported in Tables 1-4 for loadbase conditions, operation near of the MLP, considering reactive limits and heavy loading conditions besides reactive limits enforcement, respectively. It is worth to mention that the remainder cases are not shown in this table because the studied LF techniques have failed. For the sake of brevity, only results for $tol = 10^{-10}$ have been reported.

Table 1 Results obtained by proposed GN-LF and other LF techniques at loadbase conditions

	case3012wp		case3375wp		case13659pegase	
	#iter	Time [s]	#iter	Time [s]	#iter	Time [s]
Standard NR	Div.	--	Div.	--	Div.	--
Iwamoto	>100	--	>100	--	Fail*	--
4th order Runge-Kutta [2]	Div.	--	Div.	--	37	13.361
2nd order Adams-Bashforth [9]	31	0.6410	32	0.7345	23	2.3154
Heun's method [9]	63	2.3796	64	2.7384	44	4.8669
Jacobian adj. Adams-Bashforth [9]	24	0.7846	25	0.9877	9	1.6294
Reverse Bulirsch-Stoer [10]	16	0.9874	16	1.0990	18	5.2637
High order Levenberg [6]	>100	--	>100	--	>100	--
Levenberg with non-monotone line-search [7]	>100	--	>100	--	>100	--
GN-LF $\lambda^{(0)} = 10^3$	8	0.1658	8	0.1744	9	1.0491
GN-LF $\lambda^{(0)} = 10^9$	9	0.1672	9	0.1856	10	1.1651

* Low voltage solution

Table 2 Results obtained by proposed GN-LF and other LF techniques near of the MLPs

	case3012wp		case3375wp		case13659pegase	
	#iter	Time [s]	#iter	Time [s]	#iter	Time [s]
Standard NR	Div.	--	Div.	--	Div.	--
Iwamoto	>100	--	>100	--	69	6.4553
4th order Runge-Kutta [2]	Div.	--	Div.	--	Fail*	--
2nd order Adams-Bashforth [9]	35	0.6979	37	0.8210	Div.	--
Heun's method [9]	69	2.5797	70	2.9208	Fail*	--
Jacobian adj. Adams-Bashforth [9]	28	0.8960	28	1.0686	Fail*	--
Reverse Bulirsch-Stoer [10]	16	0.9874	16	1.0990	Div.	--
High order Levenberg [6]	>100	--	>100	--	>100	--
Levenberg with non-monotone line-search [7]	>100	--	>100	--	>100	--
GN-LF $\lambda^{(0)} = 10^3$	14	0.2918	14	0.3406	12	1.4760
GN-LF $\lambda^{(0)} = 10^9$	15	0.3125	15	0.3607	13	1.5947

* Low voltage solution

Table 3 Results obtained by proposed GN-LF and other LF techniques considering reactive limits

	case3012wp		case3375wp		case13659pegase	
	#iter	Time [s]	#iter	Time [s]	#iter	Time [s]
Standard NR	Div.	--	Div.	--	Div.	--
Iwamoto	>100	--	>100	--	>100	--
4th order Runge-Kutta [2]	Div.	--	Div.	--	>100	--
2nd order Adams-Bashforth [9]	Div.	--	Div.	--	Div.	--
Heun's method [9]	Div.	--	Div.	--	88	15.4032
Jacobian adj. Adams-Bashforth [9]	Div.	--	Div.	--	Div.	--
Reverse Bulirsch-Stoer [10]	Div.	--	Div.	--	36	11.1073
High order Levenberg [6]	>100	--	>100	--	>100	--
Levenberg with non-monotone line-search [7]	>100	--	>100	--	>100	--
GN-LF $\lambda^{(0)} = 10^3$	24	0.4737	32	0.7220	18	2.1317
GN-LF $\lambda^{(0)} = 10^9$	29	0.5676	36	0.8195	20	2.3550

Table 4 Results obtained by proposed GN-LF and other LF techniques considering reactive limits and heavy loading conditions

	case3012wp		case3375wp		case13659pegase	
	#iter	Time [s]	#iter	Time [s]	#iter	Time [s]
Standard NR	Div.	--	Div.	--	Div.	--
Iwamoto	>100	--	>100	--	>100	--
4th order Runge-Kutta [2]	Div.	--	Div.	--	70	25.6984
2nd order Adams-Bashforth [9]	Div.	--	Div.	--	Div.	--
Heun's method [9]	Div.	--	Div.	--	Div.	--
Jacobian adj. Adams-Bashforth [9]	Div.	--	Div.	--	Div.	--
Reverse Bulirsch-Stoer [10]	Div.		Div.	--	Div.	--
High order Levenberg [6]	>100	--	>100	--	>100	--
Levenberg with non-monotone line-search [7]	>100	--	>100	--	>100	--
GN-LF $\lambda^{(0)} = 10^3$	71	1.9085	56	1.7589	19	2.2859
GN-LF $\lambda^{(0)} = 10^9$	85	2.2570	68	2.0589	22	2.6189

On the basis of the results shown in the previous tables, the following conclusions should be remarked:

- In all studied cases, the standard NR has diverged. It is well known that the standard NR method is not suitable for solving the ill-conditioned systems, where the Jacobian matrix becomes singular.
- Levenberg's techniques [6, 7] have failed due to an excessive number of iterations. It means that despite the flat start is inside of their ROAs, it is not close enough to the solution and, therefore, these techniques are quite slow.
- Some techniques occasionally converged to the low voltage solution. Since the high voltage solution is aimed to achieve, this situation is considered a failure.

- Other techniques [2, 9, 10] have successfully converged in some cases. However, they are much less efficient than the proposed GN-LF mainly due to these techniques frequently require several matrix factorizations per iteration.
- When the loading level of the system is near of the MLP, number of iterations normally increases.
- Consideration of reactive limits provoke that solution of the LF were harder to reach. This is due to converting some PV buses to PQ during LF solutions, Consequently, size of the system grows. In fact, most of studied techniques diverge when reactive limits are taken into account.
- Despite that the studied techniques are not converged in some systems at loadbase conditions (i.e. RTE and synthetic grid models from the Texas A&M University), the proposed GN-LF approach has successfully solved all cases for all scenarios considered.

4.5.- *Performance of proposed GN-LF approach (measuring the degree of unsolvability)*

In the previous section, an algorithm for avoiding the divergence beyond solvable region is proposed. This algorithm allows to calculate an approximate solution for the LF problem where the Jacobian matrix is singular. This corresponds to an approximate solution of (5).

Fig. 8 shows the minimum value of (5) for different loading conditions in the studied cases. From this figure, it can be observed that the minimum value of SSR function drastically increases when the MLP is surpassed. Beyond the MLP, it progressively increases.

Specifically, it can be noted that the proximity to 1 and the closeness to the MLP may be directly related. Authors are currently working towards developing an effective method for calculating the MLP using the minimum of (5).

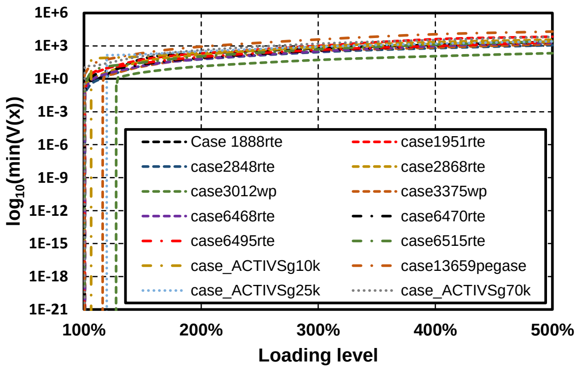

Fig 8 Minimum value of the function (5) in the studied cases for different loading conditions

4.6.- *Calculating the solution space boundary*

Further simulation studies are oriented in assessing the application of proposed GN-LF for exploring the solution space boundary. To do that, the methodology proposed in (25) and (29) is considered. As it was previously mentioned, it is necessary to notably increase the size of original LF problem in order to achieve this purpose. Therefore, we have limited our study to the 2-bus illustrative example [3].

As it was previously commented, if the system (25) is directly solved, the MLP point is calculated. In this case, taking the loadbase conditions, the MLP is obtained:

$$\begin{cases} \rho = 3.9045\,pu \\ \tan\phi_2\rho = 1.9522\,pu \end{cases} \tag{30}$$

which corresponds to the MLP for $\tan\phi_2 = 0.5$. For the sake of completeness, the elements of matrix (26) at first iteration are reported as:

$$\nabla_x \mathbf{f}(\boldsymbol{x}^{(0)}) = \begin{bmatrix} 10 & 0 \\ 0 & 30 \end{bmatrix} \tag{31}$$

$$\nabla_x\left(\nabla_x \mathbf{f}(\boldsymbol{x}^{(0)})\boldsymbol{v}^{(0)}\right) = \begin{bmatrix} -10 & 10 \\ -10 & -20 \end{bmatrix} \tag{32}$$

$$2\boldsymbol{v}^{(0)} = [2 \quad 2] \tag{33}$$

$$\boldsymbol{\Gamma}^{(0)} = [-0.5 \quad 1] \tag{34}$$

$$\boldsymbol{S}_{sch}^{(0)} = \begin{bmatrix} -1 & 0 \\ 0 & -0.5 \end{bmatrix} \tag{35}$$

The proposed methodology also allows exploring the solution space boundary. To do that, let the value of ϕ_2 be randomly changed. The solution space boundary (Fig. 9) is obtained by solving repeatedly the system (25) for different power factors.

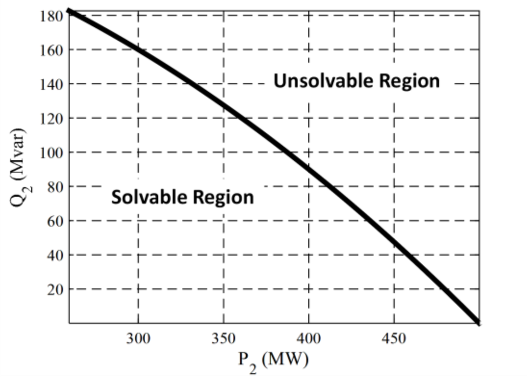

Fig 9 Solution space boundary of the 2-bus illustrative example

As it was previously commented, matrix (26) is typically very sparse, which is more noticeable in large-scale systems, however, it tends to be ill-conditioned. It had a direct impact when authors carried out the same experiment with the NR method, showing bad performance for some values of ϕ_2.

5. Conclusions

In this paper, an effective approach (GN-LF) has been proposed for solving the LF problem of ill-conditioned systems. The proposed approach has been comprehensively validated using several ill-conditioned systems in loadbase conditions, near of the maximum loadability points and considering generator' reactive limits. The results have shown the superior performance of proposed GN-LF compared with other techniques in terms of robustness and efficiency. In addition, a simple strategy for avoiding the divergence of proposed approach beyond the MLP has been developed. Finally, a methodology for determining the solution space boundary has been developed. Extensive numerical results obtained show the potential benefits of proposed GN-LF approach for different applications of power system analysis.

In the future work, the outstanding robustness of proposed GN-LF approach should be exploited for solving the hardly-convergence problems like the temperature dependent load-flow proposed in [31].

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