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Results in Mathematics

On the Rates of Pointwise Convergence for Bernstein Polynomials

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Abstract. Let f be a real bounded function defined on the interval $[0, 1]$, which is affine on $(a, b) \subset [0, 1]$, and let $B_n f$ be its associated nth Bernstein polynomial. We prove that, for any $x \in (a, b)$, $|B_n f(x) - f(x)|$ converges to 0 as $n \to \infty$ at an exponential rate of decay. Moreover, we show that this property is no longer true at the boundary of (a, b). For Bernstein–Kantorovich type operators similar properties hold, whenever f is assumed to be constant instead of affine.

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1. Introduction

Let $\mathbb{N}$ be the set of positive integers. Denote by $B[0, 1]$ the set of all real bounded functions defined on $[0, 1]$ endowed with the usual supremum norm $\| \cdot \|$, by $C[0, 1]$ the subset of continuous functions, and by $C^k[0, 1]$ the subset of k-times continuously differentiable functions. Unless otherwise specified, we assume from now on that $n, k \in \mathbb{N}$ and $x \in [0, 1]$.

Recall that the nth Bernstein polynomial of f is defined as

$$B_n f(x) = \sum_{j=0}^{n} f\left(\frac{j}{n}\right) \binom{n}{j} x^j (1 - x)^{n-j}, \quad f \in B[0, 1]. \tag{1}$$

The rates of uniform convergence of $B_n f$ towards a function $f \in C[0, 1]$, as $n \to \infty$, are characterized in terms of the so-called Ditzian–Totik second

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modulus of smoothness of f (see, for instante, [3,10,11], and the references therein).

With respect to the pointwise convergence, many different results have been obtained. Among them, we mention the following (see Păltănea [15])

$$|B_n f(x) - f(x)| \leq \frac{3}{2}\omega_2\left(f; \frac{\varphi(x)}{\sqrt{n}}\right), \quad f \in C[0,1], \tag{2}$$

where $\varphi(x) = \sqrt{x(1-x)}$ and $\omega_2(f;\cdot)$ is the usual second modulus of continuity of f defined as

$$\omega_2(f;\delta) = \sup\left\{|f(x-h) - 2f(x) + f(x+h)| : x \pm h \in [0,1],\ 0 \leq h \leq \delta\right\}, \quad \delta \geq 0.$$

Observe that the order of magnitude of the upper bound in (2) cannot be better than n^{-1}.

More recently, several authors have obtained a quantitative form of the generalized Voronovskaja's formula (see [8,16], and [7], among others). More precisely, let $\mu_{n,j}(x)$ be the jth central moment of the Bernstein polynomial, i.e.,

$$\mu_{n,j}(x) = B_n\left(e_1 - xe_0\right)^j(x), \qquad e_i(t) = t^i,$$
$$t \in [0,1],\ i \in \{0,1\},\ j \in \mathbb{N} \cup \{0\}.$$

If $f \in C^{2k}[0,1]$, then (cf. [2, Corollary 3])

$$\left|B_n f(x) - \sum_{j=0}^{2k} \frac{f^{(j)}(x)}{j!}\mu_{n,j}(x)\right| \leq \frac{1}{6^k n^k}\left(\frac{1}{k!} + \frac{(2r)!}{6^r(k+r)!}\right)\omega\left(f^{(2k)}; \frac{1}{\sqrt{n}}\right), \tag{3}$$

where $\omega(f^{(2k)};\cdot)$ is the usual first modulus of continuity of $f^{(2k)}$ and

$$r = \left\lfloor 1 + \sqrt{1 + \frac{3}{2}k}\right\rfloor,$$

$\lfloor\cdot\rfloor$ denoting the integer part. An immediate consequence of (3) is that if $f^{(j)}(x) = 0$, $j = 2,3,\ldots,2k$, then the rate of convergence of $|B_n f(x) - f(x)|$ is faster than n^{-k}. In this respect, note that the Bernstein operator leaves affine functions invariant, so it is centered, that is, $\mu_{n,1}(x) \equiv 0$. Hence, the value of $f^{(1)}(x)$ in (3) is not relevant.

On the other hand, let $D_{loc}(x_0) \subset B[0,1]$ be the set of those functions f for which $f^{(i)}(x_0) = 0$, $i \in \mathbb{N} \cup \{0\}$, for some $x_0 \in (0,1)$. In [12], it was shown that if $f \in D_{loc}(x_0)$, then $B_n f(x_0) = o(n^{-\infty})$, meaning that $B_n f(x_0) = o(n^{-k})$, for all $k \in \mathbb{N}$. This property is actually true for a large class of positive linear operators (cf. [12]).

In view of these facts, a natural question is what would be the rate of convergence of $|B_n f(x) - f(x)|$ towards 0 if f is affine on an open interval which contains x, that is to say, if there exist an interval $(a,b) \subset [0,1]$ and a function $\ell \in \operatorname{span}(\{e_0, e_1\})$, such that $x \in (a,b)$ and $f(t) = \ell(t)$, $t \in (a,b)$.

In the following section, we show that for any $f \in B[0,1]$ (not necessarily continuous) fulfilling this assumption, $|B_n f(x) - f(x)|$ has an exponential rate of decay. This property is based on accurate estimates of the tail probabilities of the binomial distribution. As related remarks, we also show that: (i) the exponents in the rates of convergence cannot be improved, in general; (ii) at the boundary of (a,b), the rate of convergence may be of polynomial order, or even no convergence may occur; and (iii) under the weaker condition that $f \in D_{loc}(x_0)$, an exponential rate of decay at x_0 may not be true. Finally, in Section 3, we show that similar properties hold for Bernstein–Kantorovich type operators, whenever f is assumed to be constant instead of affine, the reason behind this being that such operators are not centered in general.

2. Main Results

Let $Y_1(x), \dots, Y_n(x)$ be independent copies of a random variable $Y(x)$ having the Bernoulli distribution with success probability x, that is,

$$P(Y(x)=1) = 1 - P(Y(x)=0) = x. \tag{4}$$

Denote by $S_n(x) = Y_1(x) + \cdots + Y_n(x)$. Since this random variable has the binomial distribution with parameters n and x, we can rewrite (1) in probabilistic terms as

$$B_n f(x) = \mathbb{E} f\left(\frac{S_n(x)}{n}\right), \quad f \in B[0,1], \tag{5}$$

where $\mathbb{E}$ stands for mathematical expectation.

The quantity

$$r(x,\theta) = \theta \log \frac{\theta}{x} + (1-\theta) \log \frac{1-\theta}{1-x}, \quad x, \theta \in (0,1), \tag{6}$$

is called the Kullback–Leibler divergence between Bernoulli random variables with success probabilities x and θ. Arratia and Gordon [4] showed that

$$P(S_n(x) \geq bn) \leq e^{-nr(x,b)}, \quad 0 < x < b < 1. \tag{7}$$

Since the random variables $S_n(x)$ and $n - S_n(1-x)$ have the same law, we have from (6) and (7)

$$\begin{aligned} P(S_n(x) \leq an) &= P\left(S_n(1-x) \geq n(1-a)\right) \\ &\leq e^{-nr(1-x,1-a)} = e^{-nr(x,a)}, \quad 0 < a < x < 1. \end{aligned} \tag{8}$$

Denote by $\lceil y \rceil$ the ceiling of $y \in \mathbb{R}$. Recently, Ferrante [6] has shown the following refinement of (7).

Theorem A. *Let* $0 < x < b < 1$. *Suppose that*

$$n \geq 2 \qquad \text{and} \qquad 1 \leq bn \leq n-1. \tag{9}$$

Then,

$$P(S_n(x) \geq bn) \leq \frac{\beta(1-x)}{\beta - x} \frac{1}{\sqrt{2\pi\beta(1-\beta)n}} e^{-nr(x,b)}, \tag{10}$$

where $\beta = \lceil bn \rceil / n$.

Remark 1. The restriction $x < b$ in estimates (7) and (10) cannot be weakened. Otherwise, we have from the central limit theorem (see Billingsley [5, p.357])

$$\lim_{n\to\infty} P\left(S_n(b) \geq bn\right) = \lim_{n\to\infty} P\left(\frac{S_n(b) - bn}{\varphi(b)\sqrt{n}} \geq 0\right) = P(Z \geq 0) = \frac{1}{2}, \quad 0 < b < 1,$$

where Z is a standard normal random variable.

Remark 2. It is shown in [6] that the upper bound in (10) is asymptotically sharp, as $n \to \infty$. In fact, Theorem A is implicitly an asymptotic result, because, for a fixed $b \in (0,1)$, condition (9) implies that

$$n \geq \max\left(\frac{1}{b}, \frac{1}{1-b}\right).$$

Remark 3. An upper bound for the left tail probability $P(S_n(x) \leq an)$, $0 < a < x < 1$, analogous to (10), can be derived proceeding as in (8) and applying Theorem A.

In contraposition to Theorem A, no restrictions on n are required in estimates (7) and (8). For this reason, we will use such estimates in what follows.

For a fixed $x \in (0,1)$, let $f(x,\theta)$ and $g(x,\theta)$ be two real functions defined for $0 < \theta < 1$. By $f(x,\theta) \sim g(x,\theta)$, $\theta \to x$, we mean that $f(x,\theta)/g(x,\theta) \to 1$, as $\theta \to x$. Differentiating with respect to θ the function $r(x,\theta)$ defined in (6), we obtain $r(x,x) = 0$, $r^{(1)}(x,x) = 0$, and $r^{(2)}(x,\theta) = 1/\varphi^2(\theta)$. We therefore have

$$r(x,\theta) \sim \frac{1}{2}\frac{(\theta - x)^2}{\varphi^2(x)}, \quad \text{as } \theta \to x. \tag{11}$$

Denote by 1_A the indicator function of the set A. We state our first main result.

Theorem 1. *Let* $f \in B[0,1]$ *and* $0 < a < b < 1$. *Assume that there exists a function* $\ell \in span(\{e_0, e_1\})$, *such that* $f(t) = \ell(t)$, $t \in (a,b)$. *Then, we have for any* $x \in (a,b)$

$$|B_n f(x) - f(x)| \leq \|f - \ell\| \left(e^{-nr(x,a)} + e^{-nr(x,b)}\right).$$

Proof. . Let $x \in (a,b)$. Consider the function $g(t) = f(t) - \ell(t)$, $0 \le t \le 1$. Since $g(x) = 0$ and $B_n\ell(x) = \ell(x)$, we have from (5)

$$\begin{aligned} |B_nf(x) - f(x)| &= |B_n(g+\ell)(x) - (g+\ell)(x)| = |B_ng(x)| \\ &= \left| \mathbb{E}g\left(\frac{S_n(x)}{n}\right) 1_{\{S_n(x)/n \le a\}} + \mathbb{E}g\left(\frac{S_n(x)}{n}\right) 1_{\{S_n(x)/n \ge b\}} \right| \\ &\le \|f-\ell\| \left(P\left(\frac{S_n(x)}{n} \le a\right) + P\left(\frac{S_n(x)}{n} \ge b\right)\right) \\ &\le \|f-\ell\| \left(e^{-nr(x,a)} + e^{-nr(x,b)}\right), \end{aligned}$$

where the last inequality follows from (7) and (8). The proof is complete. □

In general, there is no hope to obtain a lower inequality for $|B_nf(x) - f(x)|$ in Theorem 1. Indeed, consider an antisymmetric function f around 1/2, that is, $f(t) = -f(1-t)$, $t \in [0,1]$. Suppose, in addition, that $f(t) = 0$ in $(1/2 - \delta, 1/2 + \delta)$, for some $0 < \delta < 1/2$. Since

$$\begin{aligned} B_nf\left(\frac{1}{2}\right) &= \mathbb{E}f\left(\frac{S_n(\frac{1}{2})}{n}\right) = -\mathbb{E}f\left(\frac{n - S_n(\frac{1}{2})}{n}\right) \\ &= -\mathbb{E}f\left(\frac{S_n(\frac{1}{2})}{n}\right) = -B_nf\left(\frac{1}{2}\right), \end{aligned}$$

we see that $B_nf(1/2) = f(1/2) = 0$.

For intervals containing one of the endpoints of $[0,1]$, we give the following results, whose proofs we omit since they follow the lines of that of Theorem 1.

Proposition 1. *Let $f \in B[0,1]$. Assume that there exists a function $\ell \in$ span $(\{e_0, e_1\})$, such that $f(t) = \ell(t)$, $t \in [0,b)$, $0 < b < 1$. Then, for any $x \in (0,b)$ we have*

$$|B_nf(x) - f(x)| \le \|f-\ell\| e^{-nr(x,b)}.$$

Proposition 2. *Let $f \in B[0,1]$. Assume that there exists a function $\ell \in$ span $(\{e_0, e_1\})$, such that $f(t) = \ell(t)$, $t \in (a,1]$, $0 < a < 1$. Then, we have for any $x \in (a,1)$*

$$|B_nf(x) - f(x)| \le \|f-\ell\| e^{-nr(x,a)}.$$

Concerning the previous results, some remarks are in order. For the sake of concreteness, we focus our attention on Proposition 1.

2.1. Sharpness of $r(x, b)$

Fix $k, m \in \mathbb{N}$, with $1 \le k \le m-1$, and $m \ge 2$, and let $b = k/m$. Using (6) and Stirling's approximation, we have for any $x \in (0, b)$

$$\left(B_{nm}1_{\{b\}}\right)(x) - 1_{\{b\}}(x) = \mathbb{E}1_{\{b\}}\left(\frac{S_{nm}(x)}{nm}\right)$$
$$= P(S_{nm}(x) = nk) = \binom{nm}{nk} x^{nk}(1-x)^{n(m-k)}$$
$$\sim \frac{1}{\sqrt{2nmb(1-b)}}\left(\left(\frac{x}{b}\right)^b\left(\frac{1-x}{1-b}\right)^{1-b}\right)^{nm} = \frac{1}{\sqrt{2nmb(1-b)}}e^{-nmr(x,b)},$$

as $n \to \infty$. This shows that the function $r(x, b)$ in the exponent is best possible.

2.2. Behaviour at the Boundary

Let $(h(n))_{n\ge1}$ be a sequence of nondecreasing positive real numbers such that $h(n) \to \infty$ and $h(n)/n \to 0$, as $n \to \infty$. Obviously, Proposition 1 is not meaningful for $x = b$, since $r(b, b) = 0$, as follows from (6). On the other hand, for a fixed n, Proposition 1 has interest only if

$$nr(x,b) \sim \frac{n}{2}\left(\frac{b-x}{\varphi(x)}\right)^2 \ge h(n) \quad \Longleftrightarrow \quad b - x \ge \varphi(x)\sqrt{\frac{2h(n)}{n}}, \quad b \to x,$$

as follows from (11). This and Remark 1 suggest that no similar result to Proposition 1 can be given for $x = b$.

In fact, using the convergence of moments in the central limit theorem (see, for instance, Billingsley [5, p. 338]), we have

$$\lim_{n\to\infty} \mathbb{E}\left|\frac{S_n(x) - nx}{\sqrt{n}}\right|^s = \varphi^s(x)\mathbb{E}|Z|^s, \qquad x \in (0,1), \quad s > 0, \tag{12}$$

where Z is a standard normal random variable. Denote by $y_+ = \max(0, y)$, $y \in \mathbb{R}$. Since $S_n(1/2)$ is a symmetric random variable, we have

$$\mathbb{E}\left(\frac{S_n(1/2)}{n} - \frac{1}{2}\right)_+^s = \frac{1}{2}\mathbb{E}\left|\frac{S_n(1/2)}{n} - \frac{1}{2}\right|^s, \qquad s > 0. \tag{13}$$

For any $s > 0$, consider the function $f_s \in C[0, 1]$ defined as

$$f_s(t) = \left(t - \frac{1}{2}\right)_+^s, \qquad t \in [0, 1].$$

By (12) and (13), we have

$$B_n f_s\left(\frac{1}{2}\right) - f_s\left(\frac{1}{2}\right) = \frac{1}{2}\mathbb{E}\left|\frac{S_n(1/2)}{n} - \frac{1}{2}\right|^s \sim \frac{1}{2^{s+1}}\mathbb{E}|Z|^s\frac{1}{n^{s/2}}, \quad n \to \infty.$$

In other words, we cannot have an exponential rate of decay in Proposition 1 for $x = b$. Even more, it may happen that $B_n f(b)$ does not converge to $f(b)$.

Indeed, let $g \in B[0,1]$ having right and left limits at $t \in (0,1)$, denoted by $g(t^+)$ and $g(t^-)$, respectively. Herzog and Hill [9] showed that

$$\lim_{n\to\infty} B_n g(t) = \frac{1}{2}\left(g(t^+) + g(t^-)\right).$$

Therefore, for the function $1_{[b,1]} \in B[0,1]$, we have

$$\lim_{n\to\infty} \left(B_n 1_{[b,1]}\right)(b) = \frac{1}{2} \neq 1 = 1_{[b,1]}(b).$$

2.3. Rates in the Set $D_{loc}(x_0)$

Let $f \in D_{loc}(x_0)$. As said in Sect. 1, it is known that $B_n f(x_0) = o(n^{-\infty})$. In such circumstances, it is natural to wonder if

$$|B_n f(x_0)| \leq K e^{-C(x_0)n}, \quad n \in \mathbb{N}, \tag{14}$$

for some positive constants K and $C(x_0)$. The following example gives us a negative answer.

Let $0 < \alpha < 1$. Consider the function

$$f(t) = \begin{cases} \exp(-|t-1/2|^{-\alpha}), & t \in [0,1] \setminus \{1/2\} \\ 0, & t = 1/2. \end{cases}$$

Observe that $f \in D_{loc}(1/2)$ with $f^{(i)}(1/2) = 0$, $i \in \mathbb{N} \cup \{0\}$. By Stirling's approximation, we have

$$\begin{aligned} B_{2n} f\left(\frac{1}{2}\right) &= \mathbb{E} f\left(\frac{S_{2n}(1/2)}{2n}\right) \geq f\left(\frac{n-1}{2n}\right) P(S_{2n}(1/2) = n-1) \\ &= \frac{n}{n+1} f\left(\frac{n-1}{2n}\right) \frac{(2n)!}{(n!)^2} \frac{1}{4^n} \sim \exp\left(-(2n)^\alpha\right) \frac{1}{\sqrt{\pi n}}, \quad n \to \infty. \end{aligned}$$

Since $0 < \alpha < 1$, this shows that inequality (14) cannot be true.

3. Bernstein–Kantorovich Type Operators

Let $k \in \mathbb{N} \cup \{0\}$. Let W_k be a random variable taking values in $[0,k]$ and independent of the random variables $Y_j(x)$, $j = 1, \ldots, n$, considered in Sect. 2. We define the operator

$$L_{n,k} f(x) = \mathbb{E} f\left(\frac{S_{n-k}(x) + W_k}{n}\right), \tag{15}$$

for any measurable function $f \in B[0,1]$. If $(V_j)_{j\geq 1}$ is a sequence of independent copies of a random variable V uniformly distributed on $[0,1]$ and $W_k = V_1 + \cdots + V_k$, $k \in \mathbb{N}$ ($W_0 = 0$), then the operator defined in (15) is the Bernstein-Kantorovich operator, denoted by $B_{n,k} := L_{n,k}$. Observe that $B_{n,0} = B_n$. For

$k \in \mathbb{N}$, we can write (see Mache [13], Acu et al. [1], and the references therein)

$$B_{n,k}f(x) = \sum_{j=0}^{n-k} \binom{n-k}{j} x^j (1-x)^{n-k-j} \int_0^1 \cdots \int_0^1 f\left(\frac{j+v_1+\cdots+v_k}{n}\right) dv_1 \ldots dv_k.$$

Observe that

$$L_{n,k}\left(e_1 - xe_0\right)(x) = \frac{1}{n}\mathbb{E}\left(S_{n-k}(x) + W_k\right) - x = \frac{\mathbb{E}W_k - kx}{n}.$$

This implies that the operator $L_{n,k}$ is not centered in general, i.e.,

$$L_{n,k}\left(e_1 - xe_0\right)(x) \not\equiv 0, \quad x \in [0,1].$$

For this reason, we state the local approximation result for $L_{n,k}$ in the following form.

Theorem 2. *Let $f \in B[0,1]$ be a measurable function and $0 < a < b < 1$. Assume that there exists a real constant c such that $f(t) = c$, $t \in (a,b)$, and assume that the interval $I = (na/(n-k), (nb-k)/(n-k))$ is nonempty. For any $x \in I$, we have*

$$|L_{n,k}f(x) - f(x)| \leq \|f - c\| \left(e^{-(n-k)r(x,na/(n-k))} + e^{-(n-k)r(x,(nb-k)/(n-k))}\right).$$

Proof. . Let $x \in I$. Denote

$$T_{n,k}(x) = \frac{S_{n-k}(x) + W_k}{n}. \tag{16}$$

Let $g(t) = f(t) - c$, $0 \leq t \leq 1$. From (15) and (16), we have

$$\begin{aligned}|L_{n,k}f(x) - f(x)| = |L_{n,k}g(x)| &= \big|\mathbb{E}g\left(T_{n,k}(x)\right) 1_{\{T_{n,k}(x) \leq a\}} \\ &\quad + \mathbb{E}g\left(T_{n,k}(x)\right) 1_{\{T_{n,k}(x) \geq b\}}\big| \\ &\leq \|g\| \left(P(T_{n,k}(x) \leq a) + P(T_{n,k}(x) \geq b)\right). \end{aligned} \tag{17}$$

From (16), we see that

$$b \leq T_{n,k}(x) \leq \frac{S_{n-k}(x) + k}{n}.$$

By virtue of (7), this implies that

$$P(T_{n,k}(x) \geq b) \leq P\left(S_{n-k}(x) \geq nb - k\right) \leq e^{-(n-k)r(x,(nb-k)/(n-k))}. \tag{18}$$

Again by (16),

$$a \geq T_{n,k}g(x) \geq \frac{S_{n-k}(x)}{n}.$$

By (8), this entails that

$$P(T_{n,k}(x) \leq a) \leq P\left(S_{n-k}(x) \leq an\right) \leq e^{-(n-k)r(x,an/(n-k))}.$$

This, (17), and (18) show the result. □

Similar results to Propositions 1 and 2 can be given for the operators $L_{n,k}$. Details are omitted.

Note Added in Proof

While the first version of this paper (arXiv:2411.10135) was under the reviewing process, the authors were informed by Malykhin and Ryutin, from the Lomonosov Moscow State University, that some of its contents had some intersections with their paper [14, Statement 1].

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