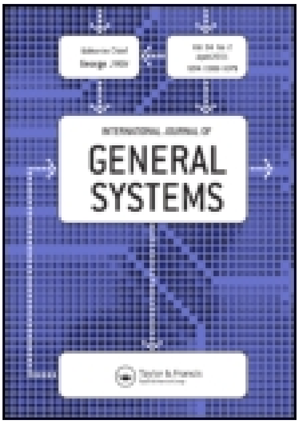

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An analysis of symbolic linguistic computing models in decision making

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An analysis of symbolic linguistic computing models in decision making

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It is common that experts involved in complex real-world decision problems use natural language for expressing their knowledge in uncertain frameworks. The language is inherent vague, hence probabilistic decision models are not very suitable in such cases. Therefore, other tools such as fuzzy logic and fuzzy linguistic approaches have been successfully used to model and manage such vagueness. The use of linguistic information implies to operate with such a type of information, i.e. processes of computing with words (CWW). Different schemes have been proposed to deal with those processes, and diverse symbolic linguistic computing models have been introduced to accomplish the linguistic computations. In this paper, we overview the relationship between decision making and CWW, and focus on symbolic linguistic computing models that have been widely used in linguistic decision making to analyse if all of them can be considered inside of the CWW paradigm.

Keywords: decision making; fuzzy linguistic approach; computing with words; symbolic linguistic computing models

1. Introduction

Real-world decision problems usually request that human beings provide either their knowledge or preferences about a set of different alternatives in a given activity to make a decision by means of reasoning processes (Evangelos 2000; Lu et al. 2007). Often these decision situations are defined under uncertain frameworks that could be managed by probabilistic models when assuming that any uncertainty can be represented by a probabilistic distribution (Martínez et al. 2009). However, it is common that uncertainty has a non-probabilistic nature. In these situations, experts feel more comfortable providing their knowledge by using terms close to human beings' cognitive model. Fuzzy logic and fuzzy linguistic approaches provide tools to model and manage such an uncertainty by means of linguistic variables (Zadeh 1975a), enhancing the flexibility, reliability of the decision models, and having provided good results in different fields (Ishibuchi et al. 2004; Martínez 2007; Pedrycz and Mingli 2011).

The use of linguistic information implies the necessity of operating with linguistic variables. Computing with words (CWW) (Zadeh 1996; Zadeh and Kacprzyk 1999a,b; Mendel et al. 2010) is defined as the methodology for reasoning, computing, and decision making whose information is described in natural language, i.e. the objects of computation are words or sentences in natural language. Therefore, it emulates human cognitive processes to improve solving processes of problems dealing with uncertainty.

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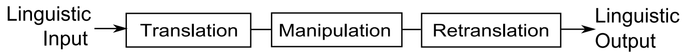

Figure 1. CWW scheme.

CWW has been and still is a key methodology in linguistic decision making (Martínez et al. 2010) not only since Zadeh (1996) coined it, but also since the early 1980s, when different researchers such as Tong and Bonissone (1980), Schmucker (1984), and Yager (1981) started to propose different computing schemes to operate with linguistic information. Such schemes are quite similar and keep a structure in which the input linguistic information should be mapped into fuzzy set models and the results should be expressed into linguistic information, which are easy to understand by human beings (see Figure 1). Yager points out the importance of the processes of translation and retranslation in CWW (Yager 1999, 2004). The former translates the linguistic inputs into a machine manipulative format based on fuzzy tools in which the computations are carried out, and the latter consists of converting the computing results into linguistic information again to facilitate the comprehension of the human beings (i.e. one of the main objectives of CWW).

Consequently, different symbolic linguistic computing models have been developed and applied as computational basis to CWW in linguistic decision making (Degani and Bortolan 1988; Herrera and Herrera-Viedma 2000; Herrera and Martínez 2000; Xu 2004; Wang and Hao 2006).

In this paper, we present a comparative study of different symbolic linguistic computing models (Herrera and Herrera-Viedma 2000; Herrera and Martínez 2000; Xu 2004; Wang and Hao 2006) widely used in linguistic decision making (Martínez et al. 2006; Chang et al. 2007; Önüt and Soner 2008; Fan et al. 2009; de Andrés et al. 2010). In this paper, we analyse their features regarding the CWW scheme as shown in Figure 1 in order to check if such computing models may be considered in the CWW paradigm.

The paper is structured as follows: Section 2 overviews CWW and its use in decision making. Section 3 reviews both linguistic and computational modellings of symbolic linguistic computing models that are most widely used in linguistic decision making. Section 4 carries out a comparative analysis among the different linguistic computational models by using a linguistic decision making problem, and finally some concluding remarks are pointed out.

2. Decision making and CWW

This section overviews the relationship between the fuzzy linguistic approach and CWW and its computational basis together with its application to linguistic decision making.

2.1 *Fuzzy linguistic approach*

In many real decision situations, the use of linguistic information is straightforward due to the nature of different aspects of the decision problem. In such situations one common approach to model the linguistic information is the fuzzy linguistic approach (Zadeh 1975a,b,c) that uses the fuzzy set theory (Zadeh 1965) to manage the uncertainty and model the information.

Zadeh (1975a) introduced the concept of linguistic variable as 'a variable whose values are not numbers but words or sentences in a natural or artificial language'. A linguistic value is less precise than a number but it is closer to human cognitive processes used to solve successful problems dealing with uncertainty. Formally, a linguistic variable is defined as follows.

DEFINITION 1 (ZADEH 1975A). A linguistic variable is characterized by a quintuple (H, $T(H)$, U, G, M), where H is the name of the variable; $T(H)$ (or simply T) denotes the term set of H, i.e. the set of names of linguistic values of H, with each value being a fuzzy variable denoted generically by X and ranging across a universe of discourse U which is associated with the base variable u; G is a syntactic rule (which usually takes the form of a grammar) for generating the names of values of H; and M is a semantic rule for associating its meaning with each H, $M(X)$, which is a fuzzy subset of U.

A linguistic variable is defined by a linguistic descriptor and its semantics. There are different approaches to choose the linguistic descriptors and different ways to define their semantics (Herrera and Herrera-Viedma 2000).

One approach for selecting descriptors consists of supplying directly the term set by considering all the terms distributed on a scale which has an order defined (Herrera et al. 1995; Yager 1995). For example, a set of seven linguistic terms S could be

$$S = \{s_0 : nothing\ \ (n), s_1 : very\ \ low\ \ (vl), s_2 : low\ \ (l), s_3 : medium\ \ (m),$$
$$s_4 : high\ \ (h), s_5 : very\ \ high\ \ (vh), s_6 : perfect\ \ (p)\}.$$

Usually in these cases, it is required the existence of the following operators:

(1) A negation operator: $Neg(s_i) = s_j$ such that $j = g - i$ ($g + 1$ is the cardinality).
(2) A maximization operator: $\max(s_i, s_j) = s_i$ if $s_i \geq s_j$.
(3) A minimization operator: $\min(s_i, s_j) = s_i$ if $s_i \leq s_j$.

The semantics of the terms is represented by fuzzy numbers defined in the interval [0,1], described by membership functions. A way to characterize a fuzzy number is to use a representation based on parameters of its membership function (Bonissone and Decker 1986). The linguistic assessments given by users are just approximate ones. Some authors consider that linear trapezoidal membership functions are good enough to capture the vagueness of these linguistic assessments (Delgado et al. 1999). The parametric representation is achieved by the 4-tuple (a, b, d, c), where b and d indicate the interval in which the membership value is 1, being a and c the left and right limits of the definition domain of the trapezoidal membership function (Bonissone and Decker 1986). A particular case of this type of representation are the linguistic assessments whose membership functions are triangular, i.e. $b = d$, so we represent this type of membership function by a 3-tuple (a, b, c). An example could be

$$perfect = (0.83, 1, 1), \quad very\ \ high = (0.67, 0.83, 1), \quad high = (0.5, 0.67, 0.83),$$
$$medium = (0.33, 0.5, 0.67), \quad low = (0.17, 0.33, 0.5), \quad very\ \ low = (0, 0.17, 0.33),$$
$$nothing = (0, 0, 0.17),$$

which is graphically shown in Figure 2.

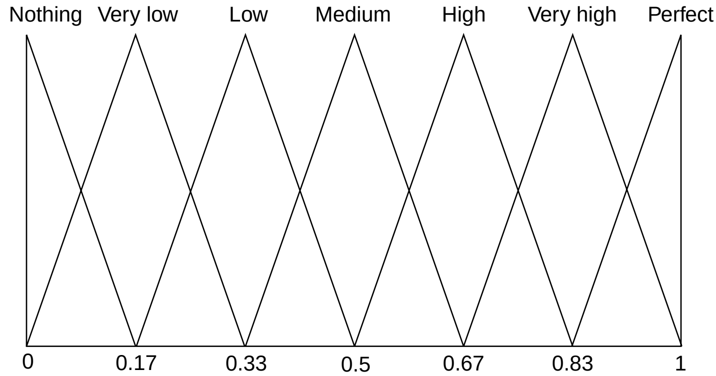

Figure 2. A set of seven terms with its semantics.

2.2 *Linguistic decision making resolution scheme*

It is necessary to analyse the phases of a linguistic decision scheme when the linguistic information is formally modelled. A common decision resolution scheme consists of two main phases (Roubens 1997):

(1) An *aggregation phase* that aggregates the values provided by the experts to obtain a collective assessment for the alternatives.
(2) An *exploitation phase* of the collective assessments to rank, sort, or choose the best ones among the alternatives.

The use of linguistic information in decision making modifies the previous scheme by introducing two new steps (Herrera and Herrera-Viedma 2000) (see Figure 3):

(1) *The choice of the linguistic term set with its semantics*. It establishes the linguistic expression domain in which experts provide their linguistic assessments about the alternatives according to their knowledge.
(2) *The choice of the aggregation operator of linguistic information*. A proper linguistic aggregation operator is chosen for aggregating the linguistic assessments. The appropriateness of the operator depends on each single decision problem.
(3) *Resolution scheme*. The best alternative/s will be chosen starting from the linguistic assessments provided by the experts, the collective assessments are then

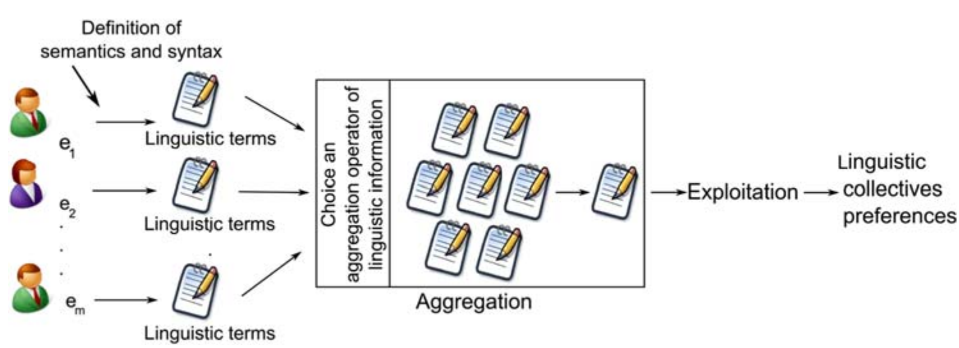

Figure 3. Scheme of a linguistic decision making problem.

computed by using the appropriate linguistic aggregation operator, and finally a ranking is obtained to choose the best alternative/s.

The previous linguistic resolution scheme shows the necessity of linguistic computing models to operate with linguistic information and obtain accurate and understandable results according to the CWW scheme. We will focus on this paper on symbolic linguistic computing models, but first we introduce some basic concepts about linguistic computing models.

2.3 *Classical linguistic computing models*

Initially, two computational models were developed to perform linguistic computations based on the fuzzy linguistic approach (Zadeh 1975a,b,c):

- *Linguistic computing model based on membership functions (semantic model).* This model carries out the computations with linguistic terms by operating with their membership functions using the Extension Principle (Dubois and Prade 1980). The Extension Principle is a basic concept in the fuzzy sets theory (Dubois and Prade 1980) which is used to generalize crisp mathematical concepts to fuzzy sets. The use of extended arithmetic based on the Extension Principle increases the vagueness of the results. Therefore, the results obtained by the fuzzy linguistic operators based on the Extension Principle are fuzzy numbers that usually do not match with any linguistic term in the initial term set. From these results we have two ways of action:
  (i) In those problems that the accuracy outweighs interpretability (ranking purposes), the results are expressed by the fuzzy numbers themselves (Anagnostopoulos et al. 2008; Fu 2008).
  (ii) If an interpretable and linguistic result is demanded, then an approximation function $app_1(\cdot)$ is applied to associate the fuzzy result $F(\mathcal{R})$ with a label in S (Degani and Bortolan 1988; Yager 2004; Martínez et al. 2006):

$$S^n \xrightarrow{\tilde{F}} F(\mathcal{R}) \xrightarrow{app_1} (\cdot) S,$$

  where S^n symbolizes the n Cartesian product of S, $\tilde{F}$ is an aggregation operator based on the Extension Principle, and $F(\mathcal{R})$ is the set of fuzzy sets over the set of real numbers $\mathcal{R}$.

  We note that the approximation process implies a loss of information and lack of accuracy of the results.
- *Symbolic linguistic computing model.* Symbolic models have been widely used in CWW because of their simplicity and high interpretability. These models use the ordered structure of the linguistic terms set, $S = \{s_0, s_1, \ldots, s_g\}$ where $s_i < s_j$ if $i < j$, to operate (Delgado et al. 1993). The intermediate results of these operations are numeric values, $\gamma \in [0, g]$, which must be approximated in each step of the process by means of an approximation function $app_2 : [0, g] \to \{0, \ldots, g\}$ that obtains a numeric value, such that, it indicates the index of the associated linguistic term, $s_{app_2(\gamma)} \in S$. Formally, it can be expressed as

$$S^n \xrightarrow{C} [0, g] \xrightarrow{app_2(\cdot)} \{0, \ldots, g\} \to S,$$

  where C is a symbolic linguistic aggregation operator, $app_2(\cdot)$ is an approximation function used to obtain an index $\{0, \ldots, g\}$ associated with a term in $S = \{s_0, \ldots, s_g\}$ from a value in $[0, g]$.

Both linguistic computing models produce loss of information due to the approximation processes and hence a lack of precision in the results. This loss of information is produced because the information representation model of the fuzzy linguistic approach is discrete in a continuous domain. Different approaches have been proposed to overcome those limitations. In the following section, we review in further detail several symbolic linguistic computing approaches that will be analysed and compared later on.

3. Symbolic linguistic computing models

Due to the relevance of linguistic decision making in real problems and the necessity of linguistic computational methods in CWW, many researchers have tried to improve the processes of CWW developing different models. This section reviews several symbolic linguistic computing models widely used in linguistic decision making, which were proposed to improve the accuracy and understandability of the processes of CWW, such as the 2-tuple linguistic model (Herrera and Martínez 2000), the virtual linguistic model (Xu 2004), and the proportional 2-tuple linguistic model (Wang and Hao 2006). This revision pays attention to the computational and representation models utilized for modelling the input information, and the results obtained for each approach in the CWW scheme are presented in Figure 1.

3.1 *2-Tuple linguistic model*

This model was presented by Herrera and Martínez (2000) to avoid the loss of information and to improve the precision in processes of CWW.

3.1.1 *Representation model*

This model represents the information by means of a pair of values (s, α), where s is a linguistic term with syntax and semantics and α is a numerical value that represents the value of the symbolic translation. Let $S = \{s_0, \ldots, s_g\}$ be a linguistic term set and $\beta \in [0, g]$ is a numerical value in its interval of granularity.

Definition 3.1 (Herrera and Martínez 2000). The symbolic translation is a numerical value assessed in $[-0.5, 0.5)$ that supports the 'difference of information' between a counting of information β assessed in the interval of granularity $[0, g]$ of the term set S and the closest value in $\{0, \ldots, g\}$ which indicates the index of the closest linguistic term in S.

This model defines a set of functions to facilitate the computational processes with 2-tuple (Herrera and Martínez 2000).

Definition 3.2 (Herrera and Martínez 2000). Let $S = \{s_0, \ldots, s_g\}$ be a set of linguistic terms. The 2-tuple set associated with S is defined as $\langle S \rangle = S \times [-0.5, 0.5)$. The function $\Delta : [0, g] \rightarrow \langle S \rangle$ is given by

$$\Delta(\beta) = (s_i, \alpha), \quad \text{with} \begin{cases} i = \text{round}(\beta), \\ \alpha = \beta - i, \end{cases} \tag{1}$$

where *round* assigns to β, the integer number $i \in \{0, 1, \ldots, g\}$ closest to β.

Proposition 3.3. Let $S = \{s_0, \ldots, s_g\}$ be a linguistic term set and (s_i, α) be a linguistic 2-tuple. There is always a function Δ^{-1} such that, from a linguistic 2-tuple, it returns its equivalent numerical value $\beta \in [0, g] \subset \mathcal{R}$.

Remark 1. From Definitions 3.1 and 3.2 and Proposition 3.3, it is clear that the conversion of a linguistic term into a linguistic 2-tuple consists in adding a value zero as the symbolic translation, $s_i \in S \Rightarrow (s_i, 0)$.

Example 3.4. Let us suppose a symbolic aggregation operation over labels assessed in $S = \{nothing, \ very\ low, \ low, \ medium, \ high, \ very\ high, \ perfect\}$ that obtains as its result $\beta = 3.25$, then the representation of this counting of information by means of a 2-tuple is (see Figure 4),

$$\Delta(3.25) = (s_3, 0.25).$$

As we can see in Figure 4, the 2-tuple linguistic model keeps the fuzzy representation and syntax according to fuzzy linguistic approach.

3.1.2 *Computational model*

A linguistic computational approach based on the functions Δ and Δ^{-1} was also defined in Herrera and Martínez (2000) with the following operations:

(1) Comparison of 2-tuple
The comparison of linguistic information represented by 2-tuple is carried out according to an ordinary lexicographic order.
Let (s_k, α_1) and (s_l, α_2) be two 2-tuple with each one representing a counting of information:
- if $k < 1$ then $(s_k, \alpha_1) < (s_l, \alpha_2)$
- if $k = 1$ then
  (a) if $\alpha_1 = \alpha_2$ then (s_k, α_1), (s_l, α_2) represents the same information
  (b) if $\alpha_1 < \alpha_2$ then $(s_k, \alpha_1) < (s_l, \alpha_2)$
  (c) if $\alpha_1 > \alpha_2$ then $(s_k, \alpha_1) > (s_l, \alpha_2)$.

(2) Negation operator of a 2-tuple
The negation operator over 2-tuple was defined as

$$Neg(s_i, \alpha) = \Delta(g - (\Delta^{-1}(s_i, \alpha))) \quad (2)$$

being $g + 1$ the cardinality of S, $S = \{s_0, \ldots, s_g\}$.

(3) 2-Tuple aggregation operators
The aggregation of information consists of obtaining a value that summarizes a set of values; therefore, the result of the aggregation of a set of 2-tuple must be a

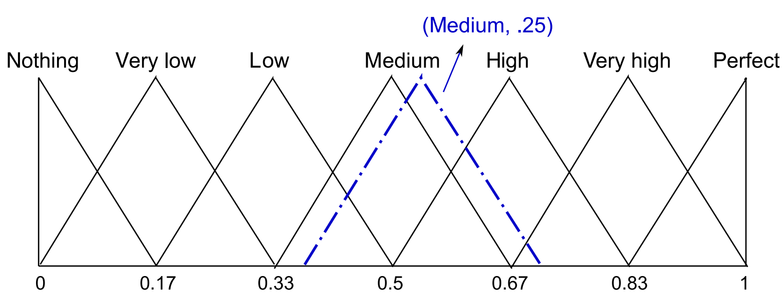

Figure 4. A 2-tuple linguistic representation.

2-tuple. There are several 2-tuple aggregation operators, such as the 2-tuple arithmetic mean, OWA, and so on (Herrera and Martínez 2000).

The 2-tuple arithmetic mean is defined as follows:

DEFINITION 3.5 (HERRERA AND MARTÍNEZ 2000). Let $x = \{(s_1, \alpha_1), \ldots, (s_n, \alpha_n)\}$ be a set of 2-tuple, the 2-tuple arithmetic mean $\bar{x}$ is computed as,

$$\bar{x} = \Delta\left(\frac{1}{n}\sum_{i=1}^{n}\Delta^{-1}(s_i, \alpha_i)\right) = \Delta\left(\frac{1}{n}\sum_{i=1}^{n}\beta_i\right). \quad (3)$$

The arithmetic mean for 2-tuple allows to compute the mean of a set of linguistic values in a precise way without any approximation process.

Example 3.6. Let us suppose an example where the expert provides his/her assessments in the linguistic term set, $S = \{s_0 : n, s_1 : vl, s_2 : l, s_3 : m, s_4 : h, s_5 : vh, s_6 : p\}$, shown in the Figure 2, providing the following linguistic preference vector:

l	h	m	vl

The aggregation of these values by using the 2-tuple arithmetic mean as aggregation operator is as follows:

- The preference vector is transformed into 2-tuple

$(l, 0)$	$(h, 0)$	$(m, 0)$	$(vl, 0)$

- The result of the aggregation is

$$\bar{x} = \Delta\left(\frac{1}{4}(\Delta^{-1}(l,0) + \Delta^{-1}(h,0) + \Delta^{-1}(m,0) + \Delta^{-1}(vl,0))\right) = \Delta(2.5) = (m, -0.5).$$

The computational model is accurate because it represents the result of the computing processes by means of a linguistic term and a numerical value that represents the value of the symbolic translation. However, it has a limitation, the labels have to be symmetrically distributed around a medium label.

3.2 *Virtual linguistic model*

This model was proposed by Xu (2004) to avoid the loss of information in processes of CWW and increase the operations in such processes.

3.2.1 *Representation model*

In this symbolic linguistic computing model, Xu extended the discrete linguistic term set S to a continuous linguistic term set $\bar{S} = \{s_\alpha | s_l < s_\alpha \leq s_t, \alpha \in [1, t]\}$, where, if $s_\alpha \in S$, s_α is called the original linguistic term, otherwise, s_α is called virtual linguistic term which does not have assigned any semantics either proper linguistic syntax.

Generally, experts use the original linguistic terms to express their assessments or preferences, and the virtual linguistic terms appear in the operations.

3.2.2 *Computational model*

Xu introduced different operational laws to accomplish the processes of CWW.

(1) Let $s_\alpha, s_\beta \in \bar{S}$ be any two linguistic terms and $\mu, \mu_1, \mu_2 \in [0, 1]$.
(2) $(s_\alpha)^\mu = s_{\alpha^\mu}$
(3) $(s_\alpha)^{\mu_1} \otimes (s_\alpha)^{\mu_2} = (s_\alpha)^{\mu_1 + \mu_2}$
(4) $(s_\alpha \otimes s_\beta)^\mu = (s_\alpha)^\mu \otimes (s_\beta)^\mu$
(5) $s_\alpha \otimes s_\beta = s_\beta \otimes s_\alpha = s_{\alpha\beta}$
(6) $s_\alpha \oplus s_\beta = s_{\alpha+\beta}$
(7) $s_\alpha \oplus s_\beta = s_\beta \oplus s_\alpha$
(8) $\mu s_\alpha = s_{\mu\alpha}$
(9) $(\mu_1 + \mu_2)s_\alpha = \mu_1 s_\alpha \oplus \mu_2 s_\alpha$
(10) $\mu(s_\alpha \oplus s_\beta) = \mu s_\alpha \oplus \mu s_\beta$

Example 3.7. Let us suppose the example presented previously to compute the arithmetic mean with this model, it is necessary to apply the above operational laws to the linguistic terms:

The arithmetic mean according to Xu is defined as

$$\bar{x}^e = \frac{\sum_{i=1}^{n} s_i}{n} = \frac{1}{n} s_{\sum_{i=1}^{n} i}. \tag{4}$$

The preference vector is aggregated and the result obtained is the following:

$$\bar{x} = \frac{1}{4} s_{(2+4+3+1)} = \frac{1}{4} s_{10} = s_{2.5}.$$

We can see that the result obtained is a virtual linguistic term that does not have assigned any semantics either proper linguistic syntax. Therefore, this computational model does not follow the CWW scheme shown in Figure 1, because the output is not linguistic.

3.3 ***Proportional 2-tuple linguistic model***

Wang and Hao (2006) presented the proportional 2-tuple linguistic model that tries to generalize and extend the 2-tuple linguistic model (Herrera and Martínez 2000). This model deals with the linguistic terms in a precise way when the support of the linguistic terms has the same width. But it does not require that the labels are symmetrically distributed around a medium label.

3.3.1 *Representation model*

This model represents the linguistic information by means of a proportional 2-tuple, such as (0.3A, 0.7B) that means a value of 30%A and 70%B. The authors remark that if B were used as the approximative grade, then some performance information would be lost. This approach is based on the concept of symbolic proportion (Wang and Hao 2006).

DEFINITION 3.8 (WANG AND HAO 2006). Let $S = \{s_0, \ldots, s_g\}$ be an ordinal term set, $I = [0, 1]$ and

$$IS \equiv I \times S = \{(\alpha, s_i) : \alpha \in [0, 1] \text{ and } i = 0, \ldots, g\}, \tag{5}$$

where S is the ordered set of $g + 1$ ordinal terms $\{s_0, \ldots, s_g\}$. Given a pair (s_i, s_{i+1}) of two successive ordinal terms of S, any two elements (α, s_i), (β, s_{i+1}) of IS is called a symbolic proportion pair and α and β are called a pair of symbolic proportions of the pair (s_i, s_{i+1}) if $\alpha + \beta = 1$. A symbolic proportion pair (α, s_i), $(1 - \alpha, s_{i+1})$ is denoted by $(\alpha s_i, (1 - \alpha)s_{i+1})$ and the set of all the symbolic proportion pairs is denoted by $\bar{S}$, i.e.

$$\bar{S} = \{(\alpha s_i, (1 - \alpha)s_{i+1}) : \alpha \in [0, 1] \text{ and } i = 0, 1, \ldots, g - 1\}.$$

Remark 2. Since for $i = \{2, \ldots, g - 1\}$, ordinal term s_i can use either $(0s_{i-1}, 1s_i)$ or $(1s_i, 0s_{i+1})$ as its representative in $\bar{S}$, by abuse of notation. $\bar{S}$ is called the ordinal proportional 2-tuple set generated by S and the members of $\bar{S}$, ordinal proportional 2-tuple, which is used to represent the ordinal information for CWW.

The representation of proportional 2-tuple allows experts to express their assessments or preferences by using two adjacent ordinals.

Similarly to the 2-tuple linguistic model (Herrera and Martínez 2000), Wang and Hao defined several functions to facilitate operations with this type of representation.

DEFINITION 3.9 (WANG AND HAO 2006). Let $S = \{s_0, \ldots, s_g\}$ be an ordinal term set and $\bar{S}$ be the ordinal proportional 2-tuple set generated by S. The function $\pi : \bar{S} \rightarrow [0, g]$ is defined by

$$\pi(\alpha s_i, (1 - \alpha)s_{i+1}) = i + (1 - \alpha), \tag{6}$$

where $i = \{0, \ldots, g - 1\}, \alpha \in [0, 1]$, and π is called the position index function of ordinal 2-tuple.

The position index function π becomes a bijection from $\bar{S}$ to $[0, g]$ and its inverse $\pi^{-1} : [0, g] \rightarrow \bar{S}$ is defined by

$$\pi^{-1}(x) = ((1 - \beta)s_i, \beta s_{i+1}), \tag{7}$$

where $i = E(x)$, E is the integer part function, and $\beta = x - i$.

This model does not have either assigned any semantics, because it uses the proportion of two consecutive linguistic terms to represent the linguistic information.

3.3.2 *Computational model*

Wang and Hao (2006) defined different operations to compute with linguistic information.

(1) Comparison of proportional 2-tuple

The comparison of linguistic information represented by proportional 2-tuple is carried out as follows:

Let $S = \{s_0, \ldots, s_g\}$ be an ordinal term set and $\bar{S}$ be the ordinal proportional 2-tuple set generated by S. For any $(\alpha s_i, (1 - \alpha)s_{i+1}), (\beta s_j, (1 - \beta)s_{j+1}) \in \bar{S}$, defines

$$(\alpha s_i, (1 - \alpha)s_{i+1}) < (\beta s_j, (1 - \beta)s_{j+1}) \Leftrightarrow \alpha i + (1 - \alpha)(i + 1)$$
$$< \beta j + (1 - \beta)(j + 1) \Leftrightarrow i + (1 - \alpha) < j + (1 - \beta).$$

Thus, for any two proportional 2-tuple $(\alpha s_i, (1 - \alpha)s_{i+1})$ and $(\beta s_j, (1 - \beta)s_{j+1})$:

- If $i < j$, then
  (a) $(\alpha s_i, (1-\alpha)s_{i+1}), (\beta s_j, (1-\beta)s_{j+1})$ represents the same information when $i = j-1$ and $\alpha = 0, \beta = 1$,
  (b) $(\alpha s_i, (1-\alpha)s_{i+1}) < (\beta s_j, (1-\beta)s_{j+1})$ otherwise.
- If $i = j$, then
  (a) if $\alpha = \beta$ then $(\alpha s_i, (1-\alpha)s_{i+1}), (\beta s_j, (1-\beta)s_{j+1})$ represents the same information,
  (b) if $\alpha < \beta$ then $(\alpha s_i, (1-\alpha)s_{i+1}) < (\beta s_j, (1-\beta)s_{j+1})$,
  (c) if $\alpha > \beta$ then $(\alpha s_i, (1-\alpha)s_{i+1}) > (\beta s_j, (1-\beta)s_{j+1})$.

(2) Negation operator of a proportional 2-tuple

The negation for proportional 2-tuple is defined as

$$Neg(\alpha s_i, (1-\alpha)s_{i+1}) = ((1-\alpha)s_{g-i-1}, \alpha s_{g-i}), \quad (8)$$

where $g + 1$ is the cardinality of S, $S = \{s_0, \ldots, s_g\}$

(3) Proportional 2-tuple aggregation operators

Different aggregation operators were defined by Wang and Hao to carry out processes of CWW. The definitions of these aggregation operators are based on canonical characteristic values of linguistic labels. To do so, they used the similar corresponding aggregation operators developed in Herrera and Martínez (2000) in order to aggregate ordinal 2-tuple through their position indexes (Wang and Hao 2006).

Example 3.10. Following the previous example applied to proportional 2-tuple linguistic model.

- The arithmetic mean according to Wang and Hao is defined as follows:

$$\bar{x} = \pi^{-1}\left(\sum_{i=1}^{n} \frac{1}{n}\pi(\alpha s_i, (1-\alpha)s_{i+1})\right) = \pi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}(i + (1-\alpha)\right). \quad (9)$$

- The preference vector is transformed into proportional 2-tuple as follows:

$$(1l, 0m) \quad (1h, 0vh) \quad (1m, 0h) \quad (1vl, 0l).$$

- The result obtained under ordinal proportional 2-tuple contexts is

$$\begin{aligned}\bar{x} &= \pi^{-1}\left(\frac{1}{4}(\pi(1l, 0m) + \pi(1h, 0vh) + \pi(1m, 0h) + \pi(1vl, 0l))\right)\\ &= \pi^{-1}\left(\frac{1}{4}(2 + 4 + 3 + 1)\right)\\ &= \pi^{-1}(2.5)\\ &= ((1-0.5)l, 0.5m)\\ &= (0.5l, 0.5m).\end{aligned}$$

4. Comparative analysis

In this section, we carry out a comparative analysis among the different symbolic linguistic computing models introduced in Section 3 by applying them to the solving process of a linguistic decision problem. The aim of such a comparative analysis is to analyse if all revised symbolic linguistic computing models can be considered in the paradigm of CWW according to the scheme shown in Figure 1.

4.1 *Solving a linguistic decision-making problem by different symbolic linguistic computing models*

Let us suppose that a conference committee $E = \{e_1, e_2, e_3, e_4\}$ compound by four scientists wants to grant a best paper award. To do so, scientists provide their opinions on the set of alternatives $X = \{x_1 = author\ 1's\ paper, x_2 = author\ 2's\ paper, x_3 = author\ 3's\ paper, x_4 = author\ 4's\ paper\}$, which are assessed by using the linguistic term set $S = \{s_0 : nothing\ (n), s_1 : very\ low\ (vl), s_2 : low\ (l), s_3 : medium\ (m), s_4 : high\ (h), s_5 : very\ high\ (vh), s_6 : perfect\ (p)\}$, (see Figure 2). Each scientist provides a preference vector (see Table 1), where $\mu_{i,j}$ is the degree of preference that the scientist, e_i, provides regarding the alternative, x_j.

Using the solving process of the linguistic decision problem presented in Section 2.2 in which the aggregation process is carried out without loss of generality by an arithmetic mean operator. Each one of the different symbolic linguistic computing models revised in Section 3 obtains the results shown in Table 2.

The fuzzy representation of the results obtained by the 2-tuple linguistic model is shown in Figure 5, whereas that the virtual linguistic model and the proportional 2-tuple linguistic model do not have any fuzzy representation. Therefore, it is not possible to represent their semantics graphically.

Even though the final solution is the same one, x_1, as can be observed in Table 2. Our aim was to analyse other features about the symbolic linguistic computing models that are further detailed in the following section.

4.2 *Comparative study*

This comparative analysis studies if the symbolic linguistic computing models revised in Section 3 and applied to the previous decision problem are based on the fuzzy linguistic

Table 1. Scientists preferences.

		Alternatives (papers)			
	μ_{ij}	*Author 1*	*Author 2*	*Author 3*	*Author 4*
Scientists	e_1	*l*	*h*	*h*	*l*
	e_2	*m*	*h*	*l*	*n*
	e_3	*h*	*vh*	*h*	*vl*
	e_4	*h*	*h*	*vh*	*l*

Table 2. Solution of the LDM problem with different symbolic linguistic computing models.

	$\bar{x}_1$	$\bar{x}_2$	$\bar{x}_3$	$\bar{x}_4$
2-Tuple	$(m, 0.25)$	$(m, -0.5)$	$(l, 0.25)$	$(m, -0.25)$
Virt. ling.	$(s_{3.25})$	$(s_{2.5})$	$(s_{2.25})$	$(s_{2.75})$
Prop. 2-tuple	$(0.75m, 0.25h)$	$(0.5l, 0.5m)$	$(0.75l, 0.25m)$	$(0.25l, 0.75m)$

Table 3. Comparative analysis of the symbolic linguistic computing models.

	2-Tuple model	*Virtual model*	*Prop. 2-tuple model*
Representation	Fuzzy	No fuzzy	No fuzzy
Com. model	Linguistic	No linguistic	Linguistic
Accuracy	Symmetrically distributed	Always, no semantic	Same width
Comprehension	Easy to understand	Useful for ordering	Understandable

approach and can be considered in the CWW paradigm, such as their authors claim. To do so, different features of the symbolic linguistic computing models regarding their representation, accuracy, and comprehension of the results are studied. Table 3 summarizes such features of the different models based on the results obtained in the decision problem solved in Section 4.1.

Therefore, we conclude regarding each analyzed feature in Table 3, the below results:

(i) Fuzzy representation
- *Virtual linguistic model*: As we can observe in Table 2, the results obtained in the linguistic decision problem by the virtual linguistic model (Xu 2004) cannot be represented in a fuzzy way, because Xu does not provide a syntax either any fuzzy semantics representation for the virtual linguistic terms. Therefore, this model is not based on the fuzzy linguistic approach (Zadeh 1975a,b,c) and cannot be considered in the CWW paradigm.
- *Proportional 2-tuple linguistic model*: The results obtained with this model (Wang and Hao 2006) have assigned a linguistic syntax, but they do not have any fuzzy representation, because they are represented by using the proportions of two consecutive linguistic terms. So, the proportional 2-tuple linguistic model cannot be either considered inside of the CWW paradigm.
- *2-Tuple linguistic model*: In Figure 5, we can see that the results obtained by 2-tuple linguistic model have assigned fuzzy semantics and syntax. Therefore, they keep a fuzzy representation of the linguistic information. Consequently, we can say that the 2-tuple linguistic model is the only one out of the three revised models that is based on the fuzzy linguistic approach and can be considered in the CWW paradigm from the fuzzy representation point of view.

(ii) Accuracy
- *Virtual linguistic model*. The computational model of the virtual linguistic model is accurate in any term set, because it does not use semantics in the term set. Besides, the results obtained in the computing processes can be values that are out of the universe of discourse of the linguistic variable.

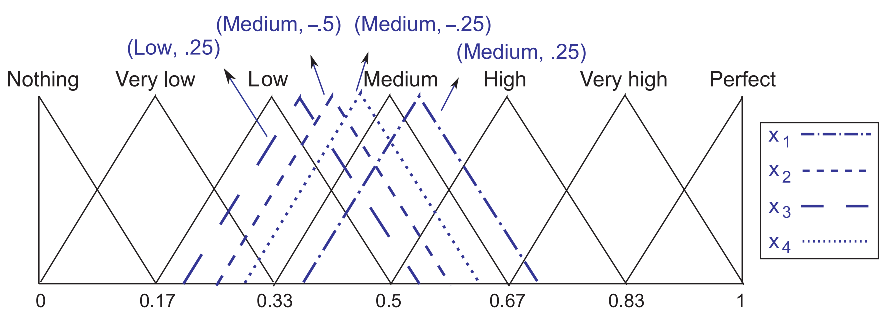

Figure 5. Results of the 2-tuple linguistic model for the linguistic decision making problem.

- *Proportional 2-tuple linguistic model*. The results obtained in the computing processes are accurate when the terms have the same width in their support. Such results just can be values that are in the universe of discourse of the linguistic variable.
- *2-Tuple linguistic model*. Its computational model is accurate when the labels are symmetrically distributed around a medium label as is shown in Figure 2. The results obtained in the computing processes just can be values that are in the universe of discourse of the linguistic variable.

(iii) Comprehension

- *Virtual linguistic model*. Analysing the results obtained in Table 2, we observe that the virtual model obtains values difficult to understand, because they are not linguistic.
- *Proportional 2-tuple linguistic model*. This model obtains qualitative results easy to understand, because they because they have assigned syntax. However, if we compare the results obtained with this model and the 2-tuple linguistic model, the results obtained with this model are a little bit more complex, because it uses four values to express just one valuation.
- *2-Tuple linguistic model*. As we can see in Table 2, the results obtained with this model are easy to understand, because they are represented by means of a linguistic term and a numerical value.

As we aforementioned, the 2-tuple linguistic model is the only model, of the revised ones, based on the fuzzy linguistic approach, since it keeps a syntax and fuzzy semantics in its representation. It is also the only one that can be considered in the CWW paradigm. Its computational model provides accurate and understandable results for human beings. For these reasons, it seems the most adequate symbolic linguistic computing model to deal with linguistic information in decision problems.

5. Conclusions

The modelling of the linguistic information to carry out the processes of CWW is crucial in decision problems. Many proposals have been provided to accomplish such processes of CWW. This contribution has reviewed different symbolic linguistic computing models, such as the 2-tuple linguistic model, the virtual linguistic model, and the proportional 2-tuple linguistic model, which have been widely applied to linguistic decision making problems, in order to carry out a comparative analysis among them.

Analysing the results obtained with such models, it has been shown that the 2-tuple linguistic model keeps the syntax and fuzzy semantics in its results, whereas the other two models do not keep them. Therefore, the 2-tuple linguistic model is the only one considered in the CWW paradigm. Additionally, its computational model provides accurate and understandable results because they are represented by means of a linguistic term and a numerical value.

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