Efficient and Reliable Power-Flow Solution Using Recursive Formula

Marcos Tostado-Véliz, Salah Kamel, and Francisco Jurado, *Senior Member, IEEE*

***Abstract*— This paper proposes a novel Power Flow (PF) solution methodology. Unlike to conventional techniques, the new approach is based on recursive formulas rather than conventional iterative methods. Stability of the novel methodology is proved and some relevant computational aspects are discussed. A wide variety of well and ill-conditioned benchmark systems shows the performance of the developed methodology. The results demonstrate that the proposed recursive formula is computationally competitive with other traditional PF methods, besides, it is scalable and reliable.**

***Index Terms*— PF analysis, Recursive formulas, Well and ill-conditioned power Systems.**

I. Introduction

A. Motivation

Many computational tools are used every day by power system operators. Among them, PF undoubtedly supposes the cornerstone. It customary finds applications in planning and operation applications, security analysis or dynamic simulations, among others [1].

Mathematically, PF is a nonlinear problem which presents several numerical challenges. Due to its importance and complexity, PF is considered a hot research topic. Recently, the authors have detected a growing interest on developing novel solution methodologies. This motivation is founded on the necessity of disposing of universal techniques, i.e. methods suitable for well and ill-conditioned systems (i.e. those cases which the Newton-Raphson (NR) fails from a flat start [2]). This implies that the new proposals should be as least as efficient as conventional techniques, but robust enough to solve ill-conditioned cases reliably. Despite the numerous recent advances experienced on this topic, the authors consider this target has not been achieved yet.

B. Literature Review

As commented, numerous works have been focused on PF analysis since its first formulations [3]. Classical solution techniques include Newton-Raphson (NR) [4], and decoupled approaches [5], among others [6]. NR and decoupled techniques are quite efficient, however, their performance in ill-conditioned cases and stiff systems (i.e. systems with heavy loading conditions or high R/X ratio) may be unsatisfactory. For circumventing these limitations, second order [7], minimization approaches [8] or sweep algorithms [9], have been proposed. The main issue of these techniques is their low degree of efficiency due to high computational burden or low convergence rate [10].

The Levenberg-type methods have recently gained popularity for solving ill-conditioned cases [11, 12]. These techniques basically consist of introducing the so-called regularization matrix, which is weighted by the damping factor with the aim of adding some elements to the diagonal of the Jacobian matrix. Thus, its condition number is enhanced and the numerical stability is improved. The main concern of these techniques is the treatment of the damping factor. Ideally, it should progressively approach zero as the algorithm converges in order to find a physic-feasible solution. This evolution should be soft in order to ensure the numerical stability of the nonlinear operator, however, if the damping factor approaches zero very slowly, the convergence of the algorithm is deteriorated, especially when the starting guess is far away from the solution. Another category of solvers which have been recently exploited for addressing ill-conditioned cases are the methods based on the Continuous Newton's framework [2], [13]-[15]. These techniques arise from the analogy between NR and the Forward-Euler method. If one considers this analogy extensible to any other numerical integration routine, a plethora of PF solvers could be developed. These techniques are wider convergent than NR but much less efficient.

The Holomorphic Embedding method has emerged as a novel technique for PF analysis [16], [17], estimating the loadability margin [18] or calculating all type-1 solutions of the PF equations [19]. This approach replaces the nonlinear operators of the conventional solvers by pure algebraic calculations, however, it still involves some heavy operations like convolutions [16]. A unified framework for universality applying this technique may be considered an open topic. In addition, its performance in ill-conditioned systems might be questionable [20].

The High Order Newton-like methods are a family of nonlinear solvers with high convergence rate (i.e. higher than two). They had not attracted much attention for PF analysis until Derakhshandeh and Pourbagher [21], and posteriorly the

M. Tostado-Véliz and F. Jurado are with Department of Electrical Engineering, University of Jaén, 23700 EPS Linares, Jaén, Spain (e-mail: mtostado@ujaen.es and fjurado@ujaen.es). S. Kamel is with the Department of Electrical Engineering, Faculty of Engineering, Aswan University, 81542 Aswan, Egypt. S. Kamel is currently with State Key Laboratory of Power Transmission Equipment & System Security and New Technology, Chongqing University, Chongqing, China 400030 (email: skamel@aswu.edu.eg).

authors [22], recently proposed various PF solvers based on High Order Newton-like methods. This kind of techniques are simple and may be computationally competitive with NR. In addition, as based on NR, they can easily incorporate alternative PF formulations and standard mechanisms for taking into account functional limits such as in [23]. Undoubtedly, the main limitation of the High Order Newton-like methods as applied to PF analysis is their performance in ill-conditioned systems. In such cases, this kind of techniques are presumably weaker than NR due to their strongly local convergence features [24]. Consequently, the starting guess should be carefully selected closer to the solution for ensuring the convergence, which is not always possible.

The PF equations are normally raised on power injections real-based form. Usually, the polar coordinates are assumed to be more practical and computationally attractive [25], while the rectangular coordinates typically find application for some techniques as in [8]. On the other hand, alternative PF formulations have been proposed in [26]-[29]. In [26], a hybrid formulation is explored, which is based on combining the current-based formulation [30], and the conventional power mismatch form. The key idea lies is to formulate the equations corresponding to PQ buses in current injections while the PV buses are considered in power injections. Thus, one takes advantage of the main features of both schemes; on the one hand, using the current-injection based approach in rectangular coordinates, some of the elements of the Jacobian matrix are constant and, consequently, they do not need to be updated, on the other hand, the formulation of the PV buses allows to reduce the size of the system. In [27], the PF equations are formulated in d-q coordinates, which is useful for some applications as those systems with a large number of electronics devices or machines. In [28], various PF formulations are studied and other variants are introduced and analysed. The complex formulation is admitted to be a more natural representation of the PF equations. It has been considered for developing robust solvers [29] or hybrid formulations [31]. Nevertheless, the real form of the PF equations still shows better results in practical applications [28].

C. *Contributions & Paper Organization*

As deduced from the Literature Reviews, most of available PF solvers present important issues, for example, related with their computational efficiency, reliability or versatility, among others. Consequently, it is difficult to find a solution method with universal character. This technique should, at least, meet the following requirements:

- The new proposal should be computationally competitive with the conventional solvers like NR. In this regard, since the factorization of the Jacobian matrix is considered as the heaviest part of a PF calculation, the novel method should require only one LU factorization each iteration (as NR).
- The new method should be more reliable than NR for being used in ill-conditioned systems. In addition, its performance should not be much affected in stiff problems.
- Some methods [9] and formulations [30], pose difficulties for managing with PV buses. The novel technique should be versatile enough for easily incorporating PV buses along with other practical functionalities such as the reactive limits or control variables [23].
- This new method should be scalable and adaptable for both distribution and transmission networks.
- Conceptually simple for facilitating its implementation in commercial software, industry tools or education and research applications.

Among all reviewed solvers, undoubtedly those based on the Newton's method such as [2], [13]-[15], [21], [22] in power injections formulation are the most suitable for meeting the requirements above. This paper proposes a novel PF solver which arises from posing the PF solution by NR as a recursive formula. This approach allows to alleviate the computational burden of NR using two recursivities (main and secondary). The new proposal is efficient and stable. In addition, it works acceptably in well and ill-conditioned systems along to be computationally tractable for any network. A potential failure of the proposed methodology, along a reliable countermeasure are discussed and explored. Several numerical experiments in various realistic cases are achieved to show the performance of the proposed PF methodology.

Remainder of this paper is organized as follows. The proposed recursive formula and its stability are presented in Section II. Section III shows a potential failure of the proposed methodology and proposes a solution. The practical computational implementation of proposed methodology is considered in Section IV. Section V presents various numerical experiments. Finally, the main conclusions are exposed in Section VI.

II. PF Solution using Recursive Formula

A. *Background*

In their most general form, the PF equations in power injections formulation with polar coordinates are established as a set of n nonlinear algebraic equations as follows:

$$\mathbf{g}(\boldsymbol{x}) = \mathbf{0} \tag{1}$$

where, $\mathbf{g}\colon \mathbb{R}^n \to \mathbb{R}^n$ are the PF equations and $\boldsymbol{x} \in \mathbb{R}^n$ are the unknowns. In PF analysis, different kind of buses are considered as a function of the variables and equations provided:

- Slack bus: its voltage angle and magnitude are known, so it does not provide any equation.
- PQ buses: which the injected active and reactive power are known and the voltage angle and magnitude are unknown. Each PQ bus provides two equations.
- PV buses: which the injected active power and the voltage magnitude are known, and the voltage angle and the injected reactive power are unknown. Each PV bus provides one equation.

Therefore, the vector of unknowns is formed by the voltage angles and magnitudes of PQ buses and the voltage angles of PV buses. The size of (1) in power injection formulation with polar coordinates is, consequently, $n = 2n_{\mathrm{PQ}} + n_{\mathrm{PV}}$ (where n_{PQ} is the number of PQ buses and n_{PV} is the number of PV buses).

One of the most widely employed iterative technique for solving (1) is NR, which proceeds as follows:

$$\boldsymbol{x}^{i+1} = \boldsymbol{x}^i - [\mathbf{J}(\boldsymbol{x}^i)]^{-1}\mathbf{g}(\boldsymbol{x}^i) \tag{2}$$

where, $\mathbf{J} \in \mathbb{R}^{n\times n}$ is the Jacobian matrix of (1) and superindexes denote the iteration counter. Mapping (2) needs a LU decomposition along a linear system solution each iteration, moreover, the convergence of this iterative procedure is eminently local. It means that the mapping (2) only converges if the starting guess (i.e. $\boldsymbol{x}^0$) lies inside of the so-called Region of Attraction. The Region of Attraction of a nonlinear solver is defined as the set of initial points which the technique converges, i.e., if a starting point lies inside of the Region of Attraction, the technique will converge, otherwise, the solution is not reachable using this solver. NR and the High order Newton-like methods are characterized for a narrow Region of Attraction, which limits their application in ill-conditioned systems, where the Region of Attraction is normally narrower than for well-conditioned problems. Certainly, one may shall to consider a solver with global convergence, however, this is difficult to obtain in practise and pose serious difficulties in some cases. So that, it is usually admitted to get wider convergence, at least acceptable to offer good performance in most of ill-conditioned cases [2]. Limitations of mapping (2) have motivated a constant search of alternative methodologies. In this paper, we propose an approach by posing (2) as a recursive (or semicursive) formula as described in the following Section.

B. *PF Solution Using Recursive Formulas*

Firstly, it is important to note that for the mapping (2) one has:

$$\boldsymbol{x}^{\infty} = \boldsymbol{x}^{\star} \tag{3}$$

where, $\boldsymbol{x}^{\star}$ denotes a PF solution (i.e. $\mathbf{g}(\boldsymbol{x}^{\star}) = \mathbf{0}$). Eq. (3) indicates that $\boldsymbol{x}^{\star}$ is a fixed point of the mapping (2). One should notice that:

$$\boldsymbol{x}^{\star} = \mathbf{f}(\boldsymbol{x}^{\infty}) = \mathbf{f}\left(\mathbf{f}(\dots\mathbf{f}(\boldsymbol{x}^0))\right) \tag{4}$$

where, $\mathbf{f}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a function which defines a fixed point iteration, e.g. the mapping (2) or any other PF method.

It is interesting to note that the eminent recursive character of (4). Here, the value of $\boldsymbol{x}^i$ is a function of $\boldsymbol{x}^{i-1}, \boldsymbol{x}^{i-2}, \dots, \boldsymbol{x}^0$. Of course, equation (4) could be also written in iterative form. Thus, it is possible to pose a specified iterative solver in its recursive counterpart; for example, let us define $\mathbf{f}^{i+1} = \mathbf{f}^i - [\mathbf{J}(\mathbf{f}^i)]^{-1}\mathbf{g}(\mathbf{f}^i)$, thus, one obtains the iterative mapping (2) if $\mathbf{f}^i = \boldsymbol{x}^i$ and one easily could write (2) in the form (4). In this example, it is important to note that $\mathbf{f}^i$ is still a function of $\mathbf{f}^{i-1}, \mathbf{f}^{i-2}, \dots, \mathbf{f}^0$; thus, the solution of the PF problem will be reached for the ∞^{th} recursivity, and it may be written as:

$$\boldsymbol{x}^{\star} = \mathbf{f}^{\star} = \mathbf{f}^{\infty} = \mathbf{f}\left(\mathbf{f}(\dots(\mathbf{f}^0))\right) \tag{5}$$

Of course, the ∞^{th} recursivity could not be reached in practise, however, as for iterative mappings, one could also stop at that i^{th} which a required convergence tolerance is achieved.

C. *Formulation with p Recursivities*

Let us consider the case which two recursivities are involved as follows:

$$\begin{cases} \mathbf{f}^i = \mathbf{f}\left(\mathbf{f}(\dots(\mathbf{f}^0))\right) \\ \mathbf{f}_k^i = \mathbf{f}\left(\mathbf{f}\left(\dots(\mathbf{f}_0^i)\right)\right) \end{cases} \tag{6}$$

As observed, the paradigm (6) involves two recursivities denoted by i and k. It is important to notice that any $\mathbf{f}_k^i$ may be defined as a function of $\mathbf{f}_{k-1}^i, \mathbf{f}_{k-2}^i, \dots, \mathbf{f}_0^i$; while $\mathbf{f}^i$ is function of $\mathbf{f}^{i-1}, \mathbf{f}^{i-2}, \dots, \mathbf{f}^0$. It is important to note that the recursivity defined by k is not explicitly function of $\mathbf{f}^{i-1}, \mathbf{f}^{i-2}, \dots, \mathbf{f}^0$; i.e. it is only an explicit function of $\mathbf{f}^i$. Therefore, one can intuitively distinguish a main recursivity defined by i and a secondary recursivity defined by k. Formulation (6) could be extended to p secondary recursivities as follows:

$$\begin{cases} \mathbf{f}^i = \mathbf{f}\left(\mathbf{f}(\dots(\mathbf{f}^0))\right) \\ \mathbf{f}_{k_1}^i = \mathbf{f}\left(\mathbf{f}\left(\dots(\mathbf{f}_0^i)\right)\right) \\ \mathbf{f}_{k_1,k_2}^i = \mathbf{f}\left(\mathbf{f}\left(\dots(\mathbf{f}_{k_1,0}^i)\right)\right) \\ \vdots \\ \mathbf{f}_{k_1,k_2,\dots,k_p}^i = \mathbf{f}\left(\mathbf{f}\left(\dots(\mathbf{f}_{k_1,k_2,\dots,0}^i)\right)\right) \end{cases} \tag{7}$$

It is important to note that no matter the number of recursivites involved, the unique degree of freedom is the root of the recursivity (namely $\mathbf{f}^0$), which, for PF calculation may be defined as the starting guess $\boldsymbol{x}^0$. Therefore, the concept of Region of Attraction for our proposal could be also defined as the set of roots which the developed method converges. It is worth noting that this concept is comparable with the conventional definition of Region of Attraction as the root, in our case, is equal to the starting guess, so that, our method is comparable with the conventional solvers.

D. *Proposed Recursive Formula for PF Calculation*

In this paper, we propose the following recursive formula for solving the set of equations (1):

$$\mathbf{f}_{k+1}^i = \mathbf{f}_0^i - h_k\left[\mathbf{J}(\mathbf{f}_0^i)\right]^{-1}\sum_{j=0}^{k}\mathbf{g}(\mathbf{f}_j^i) \tag{8}$$

where, $h \in \mathbb{R}^+$ is a step size and $\mathbf{f}_0^i = \boldsymbol{x}^i$. It is essential noticing that (8) in fact involves two recursivities. On the one hand, we can differentiate a loop which the Jacobian matrix is updated. This recursivity is denoted by the superindex i. By analogy with (6), we clearly differentiate a the main recursivity defined by i, where the Jacobian matrix is updated. On the other hand, we have included a secondary recursivity which the Jacobian matrix is kept constant. This loop is denoted by the subindex k.

At this point, an obvious question arises: *which criteria is considered for "jumping" from the secondary to the main recursivity?*. At this point, we simply consider that a main recursivity is computed after $N \in \mathbb{N}$ successive secondary loops (see Section IV for further explanation). Regarding the step size, it can be updated each secondary recursivity using the following simple rule:

$$h_k = \max\left(0.5, \min\left(\left\|\left[\mathbf{J}(\mathbf{f}_0^i)\right]^{-1} \sum_{j=0}^{k} \mathbf{g}\left(\mathbf{f}_j^i\right)\right\|_\infty, 1\right)\right) \quad (9)$$

As observed, the step size is updated inversely proportional to the updating vector, which looks logic in order to preserve the reliability of the mapping (8). In addition, it has been in this case bounded to $[0.5, 1]$, for simply preserving the numerical stability. At this point, it is worth noticing that $\boldsymbol{x}^\star$ is a fixed point of (8) since $\mathbf{f}^\infty = \boldsymbol{x}^\star$. Therefore, this recursion actually constitutes itself a PF solver.

E. Why two Recursivities?

According to the framework (7), any number of different recursivities could be considered. However, our proposal (8) just includes two. In this Section, we aim at explaining why we have included two recursivities within the recursion (8). To explain that, let us consider the IEEE 30-bus system [32] as an example. We have used the recursivity formula (8) for solving it from a flat start. Fig. 1 shows the convergence profiles for different values of N. Clearly, the main recursivity has the effect of increasing the convergence rate and, consequently, accelerating the convergence. However, a LU factorization is computed each main loop. One should know that this calculation can be considered the heaviest computation of (8). On the other hand, the higher N the more number of recursivities necessary to achieve the required convergence tolerance. This is due to the secondary recursivity in (8) has lower convergence rate than a main recursivity, because the Jacobian is kept constant. Therefore, we can establish that each recursivity has a specific task. While the main recursivity is devoted on avoiding too many loops, the secondary recursions aim at alleviating the computational burden of the proposed iterative procedure. Hence, in this case, it is not justified to include more type of recursivities within the algorithm defined by (8).

The main merit will be to obtain an optimal balance among the total amount of main and secondary recursivities computed, in order to obtain a good performance from the proposed PF methodology based on recursion (5). This point will be further discussed in Section IV.

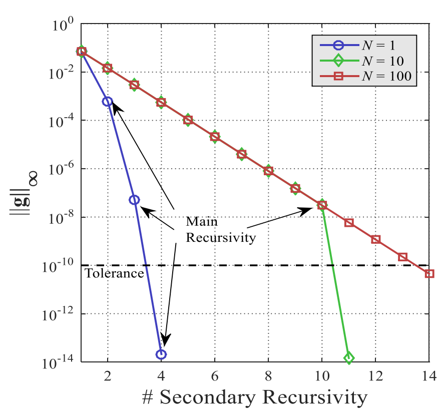

Fig. 1 Convergence profiles for different values of N (IEEE 30-bus system)

F. Stability

In this Section, we prove the numerical stability of (8). To do that, it is suitable to firstly introduce the following definition [33].

Definition 1. *An equilibrium point $\boldsymbol{x}^\star$ of $\boldsymbol{F} = \mathbf{f}_k^i - \mathbf{f}_0^i$ is called hyperbolic if the associated Jacobian of $\boldsymbol{F}$ at $\boldsymbol{x}^\star$ has no eigenvalues with a zero real part. In addition, $\boldsymbol{x}^\star$ is asymptotically stable (or sink), if all the eigenvalues of the associated Jacobian of $\boldsymbol{F}$ at $\boldsymbol{x}^\star$ has no eigenvalues with a positive real part.*

The following Theorem is devoted with the aim of demonstrating the stability of function (5).

Theorem 1. *Let $\boldsymbol{x}^\star$ be a PF solution such that $\mathbf{g}(\boldsymbol{x}^\star) = \mathbf{0}$, then, this point is asymptotically stable for the function (8).*

Proof. The function $\boldsymbol{F}$ for the recursion (8) is defined as follows:

$$\boldsymbol{F} = -h[\mathbf{J}(\mathbf{f}_0)]^{-1} \sum_{j=0}^{k} \mathbf{g}(\mathbf{f}_j) \quad (10)$$

By differentiating (10) with respect to $\boldsymbol{x}$, one obtains:

$$\nabla_x \boldsymbol{F} = -h\left(\nabla_x[\mathbf{J}(\mathbf{f}_0)]^{-1} \sum_{j=0}^{k} \mathbf{g}(\mathbf{f}_j) + [\mathbf{J}(\mathbf{f}_0)]^{-1} \nabla_x \sum_{j=0}^{k} \mathbf{g}(\mathbf{f}_j)\right) \quad (11)$$

As preceding subsection, If we consider that a main recursivity is computed each N secondary loops, equation (11) at $\boldsymbol{x}^\star$ is given by:

$$\nabla_x \boldsymbol{F}|_{x^\star} = -h(\mathbf{0} + N\boldsymbol{I}) = -hN\boldsymbol{I} \quad (12)$$

where, $\boldsymbol{I} \in \mathbb{R}^{n \times n}$. This completes the proof. □

Of course, the algorithm (8) still presents local convergence. Therefore, reachability of $\boldsymbol{x}^\star$ depends on the root chosen, which, should lie inside of the Region of Attraction. Unfortunately, there is not an analytical procedure for calculating the Region of Attraction [10]. Hence, it has to be empirically proved. This is made in the Results Section.

III. THE SANITARY STEP AND ITS IMPORTANCE

A potential failure of the recursion (8) may occur if too many secondary recursivites are successively computed. Let us consider the 3012-bus snapshot of the Polish Transmission System [33] for illustrating this point. Fig. 2 plots the convergence profile of the algorithm (8) for a sufficiently large value of N. As observed, convergence pattern of the proposed recursion formula is oscillatory, which leads to divergence.

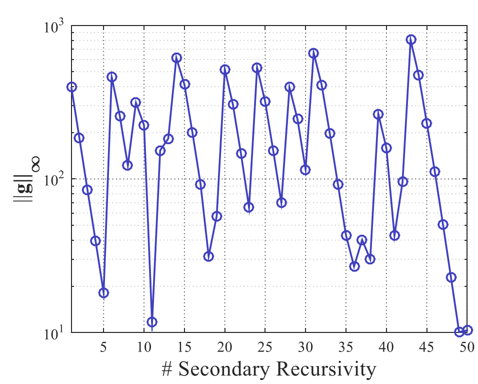

Fig. 2 Failure of proposed recursion formula (5) in the 3012-bus snapshot of the Polish Transmission System

In order to avoid this potential issue, we propose to use the following sanitary step:

$$\textbf{if } \left\|\mathbf{g}\left(\mathbf{f}_{k+1}^{i}\right)\right\|_{\infty} > \left\|\mathbf{g}\left(\mathbf{f}_{k}^{i}\right)\right\|_{\infty} \textbf{ then}$$
$$\mathbf{f}^{i+1} = \mathbf{f}_{k}^{i} \;\; \textbf{break} \tag{13}$$

Using (13), one avoids the potential failure previously showed, since $\mathbf{f}_k^i$ is replaced by $\mathbf{f}_{k-1}^i$ which is considered secure. In addition, the secondary loop is breaking in order to compute a main recursivity, which provides a better approximation of the unknown vector since the Jacobian matrix is updated. Fig. 3 is homologue to Fig. 2 but using the sanitary step (13). As clearly observed, the failure shown in Fig. 2 is now avoided.

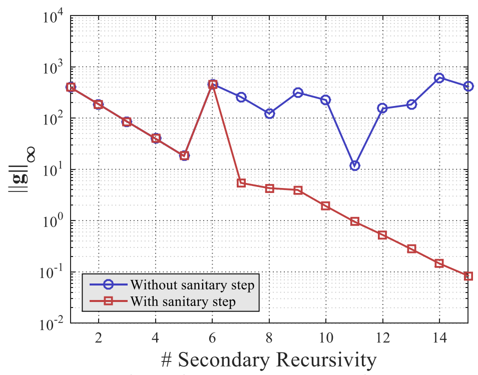

Fig. 3 Convergence profiles of proposed recursion formula (8) in the 3012-bus snapshot of the Polish Transmission System with and without the sanitary step (13)

IV. Computational Implementation

From the preceding Sections, it is clear that the recursion formula (8) implies two recursivities. As discussed, the performance of the developed PF solver strongly depends on how we alternate between the main and secondary recursivites. As commented, while the main recursivity is considered more accurate and reliable, the secondary loop is computationally cheaper. Therefore, one can guess that we should compute as secondary steps as possible. However, as showed in Sections II.B and III, too many successive secondary steps may imply an undesirable performance. Based on our experience, we have developed a functional algorithm which is devoted on applying the recursion formula (8) for PF solution. This algorithm has been conceived on the basis of the following ideas:

- At the beginning of the iterative procedure, a main recursivity has to be computed in order to calculate $\mathbf{J}\left(\mathbf{f}_0^i\right)$ which is necessary for executing the secondary steps.
- Automatically the algorithm should "jump" to the secondary loop, which is more efficient.
- The algorithm will remain in the secondary loop until $k = N$ or the sanitary step (13) is activated.
- Then, a main recursivity is computed and the process is repeated until convergence.

The flowchart for using the recursion formula (8) for PF solution on the basis of the ideas previously exposed, is sketched in Fig. 4. In addition, a practical computational implementation of the proposed PF methodology is summarized in the Algorithm 1 using pseudocode. Here, the value of $\|\mathbf{g}\|_\infty$ has been considered as convergence criterion. In addition, the algorithm is considered failed if the main recursivity counter i surpasses a predefined threshold namely $i_{\max}$. This is introduced since, as commented, the convergence of (8) is eminently local due to it is still based on the Newton's increment vector, so that, its convergence depends on the root $\mathbf{f}^0$. If the root does not lie inside of the Region of Attraction, the developed algorithm diverges and, consequently, it should be included a criterion for stopping it if that occurs.

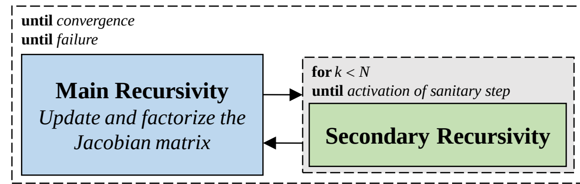

Fig. 4 Flowchart of practical implementation of proposed PF methodology based on the recursion formula (8)

Algorithm 1: Developed PF solver based on (8)

```
Let x^0, ε, i_max and N be given
Initialize the main recursivity counter i = 0
Select the recursion root f^0 = x^0
while ||g(f^i)||_∞ > ε do
    Initialize secondary recursivity counter k = 0
    Calculate J(f_0^i) and factorize it
    for k < N do
        h_k ← Solve (9)
        f_{k+1}^i ← Solve (8)
        if ||g(f_{k+1}^i)||_∞ < ε then
            break # Convergence
        end if
        if ||g(f_{k+1}^i)||_∞ > ||g(f_k^i)||_∞ then # Sanitary step
            f^{i+1} = f_k^i
            break
        end if
        k ← k + 1
    end do
    i ← i + 1
end do
return solution x = f_{k+1}^i
```

V. Numerical Results

A. Studied Cases

During this Section, several systems are used for showing the performance of proposed PF methodology based on the formula (8):

- 30-, and 118-bus IEEE test systems [31].
- 69-bus radial test system [33].
- A Synthetic 500-bus grid [31].
- The 1354-, 9241-, and 13659-bus portions of the European Transmission System from the EU Pegase Project [34, 35].
- The 3012-bus snapshot of the Polish Transmission System at 2007-08 winter evening peak [33].

All considered cases are available within Matpower database [33]-[37].

B. Computational Efficiency

The first simulations are devoted on testing the efficiency of the proposed PF methodology. To do that, the computation time has been used as index. In this regard, all simulations have been run on a 64-bit i5-9400F Intel Core personal computer (2.90 GHz, 8 GB of RAM). In order to avoid the influence of other computational activities, the reported results have been calculated as the average values of 100 simulations.

The proposed PF methodology detailed in the Algorithm 1 has been compared with the standard NR in polar coordinates and the Optimal Multiplier method in polar coordinates (OMP) [38]. The proposed methodology is studied for $N = 5$, $N = 10$ and $N = 20$.

Fig. 5 shows a comparison of the solution times consumed by the studied PF methods for solving the considered systems from a flat start with $\varepsilon = 10^{-6}$. The comparison is reported as the relative time with respect to OMP, which was the slowest method in all studied cases. Firstly, it is worth mentioning that NR failed in the 3012-, and 13659-bus cases, therefore, according to the definition provided in [2], these systems can be categorized as ill-conditioned. Since our proposal even converged in such cases, we can assert that the recursion (8) is wider convergent than NR and, therefore, its Region of Attraction would be generally wider than that showed by NR. As observed, the proposed PF methodology was faster than OMP and NR in all tested cases. The influence of the parameter N is difficult to discern and it looks very case-dependent. Normally, the highest N entails the most competitive execution time since more secondary steps, which are very efficient, are successive computed and some main recursivities can be avoided. However, there are clear exceptions. In some cases, this is due to many secondary recursivities have to be computed for converging as explained in Section II.B. However, there are other situations, for example, in some cases a large N does not necessarily imply less main recursivities (see Section V.C).

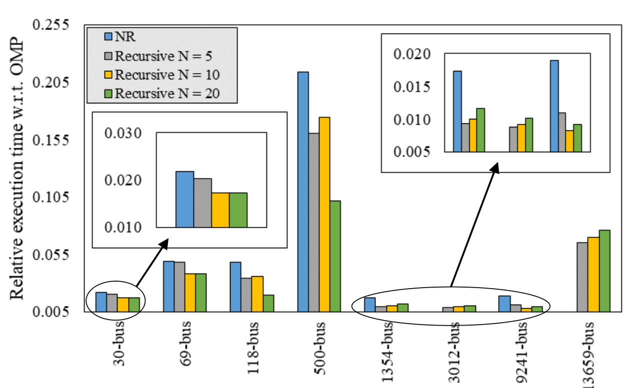

Fig. 5 Relative execution time of proposed PF methodology and NR with respect to OMP

The proposed PF methodology have been also tested in case of considering the reactive limits of generators. The strategy described in [22], has been implemented in the studied PF methods. Fig. 6 is analogue to Fig. 5 for this scenario. As observed, similar conclusions as for the preceding simulations can be drawn in this case.

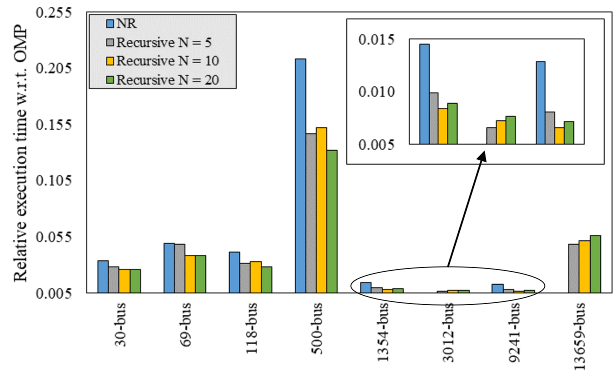

Fig. 6 Relative execution time of proposed PF methodology and NR with respect to OMP with reactive limits

C. Convergence Rates

The convergence rates of PF methods have been studied. The proposed PF method may be difficult to be compared with other techniques, due to it implies recursions rather than iterations. From a computational point of view, the main recursivity of (8) is similar to an iteration of NR and OMP. Tables I and II provide the iterations number for NR and OMP, along with the number of main and secondary recursivities required by the proposed PF technique, with and without consideration of reactive limits, respectively, from a flat start with $\varepsilon = 10^{-6}$. While NR failed in the 3012-, and 13659-bus cases, OMP and the Recursive formula (8) successfully converged in all the tested cases. It is worth mentioning that OMP converged to the low voltage solution in the 13659-bus system. If we consider the main recursivities of (8) comparable to an iteration of NR or OMP, we can see that the proposed recursive PF methodology presented higher converge rate than NR and OMP. Logically, the larger N implies the lower number of main recursivities. However, more secondary steps are normally computed for large values of N.

TABLE I
CONVERGENCE RATES WITHOUT REACTIVE LIMITS

	NR	OMP	Recursive – Main (Secondary) recursivities		
			$N = 5$	$N = 10$	$N = 20$
30-bus	3	136	2 (6)	1 (8)	1 (8)
69-bus	3	62	2 (6)	1 (7)	1 (7)
118-bus	4	81	2 (7)	2 (11)	1 (15)
500-bus	4	18	2 (7)	2 (11)	1 (13)
1354-bus	5	283	2 (8)	2 (12)	2 (21)
3012-bus	Fail	422	3 (14)	3 (18)	3 (27)
9241-bus	6	317	3 (11)	2 (13)	2 (22)
13659-bus	Fail	54*	3 (13)	3 (21)	3 (31)

** Low Voltage Solution*

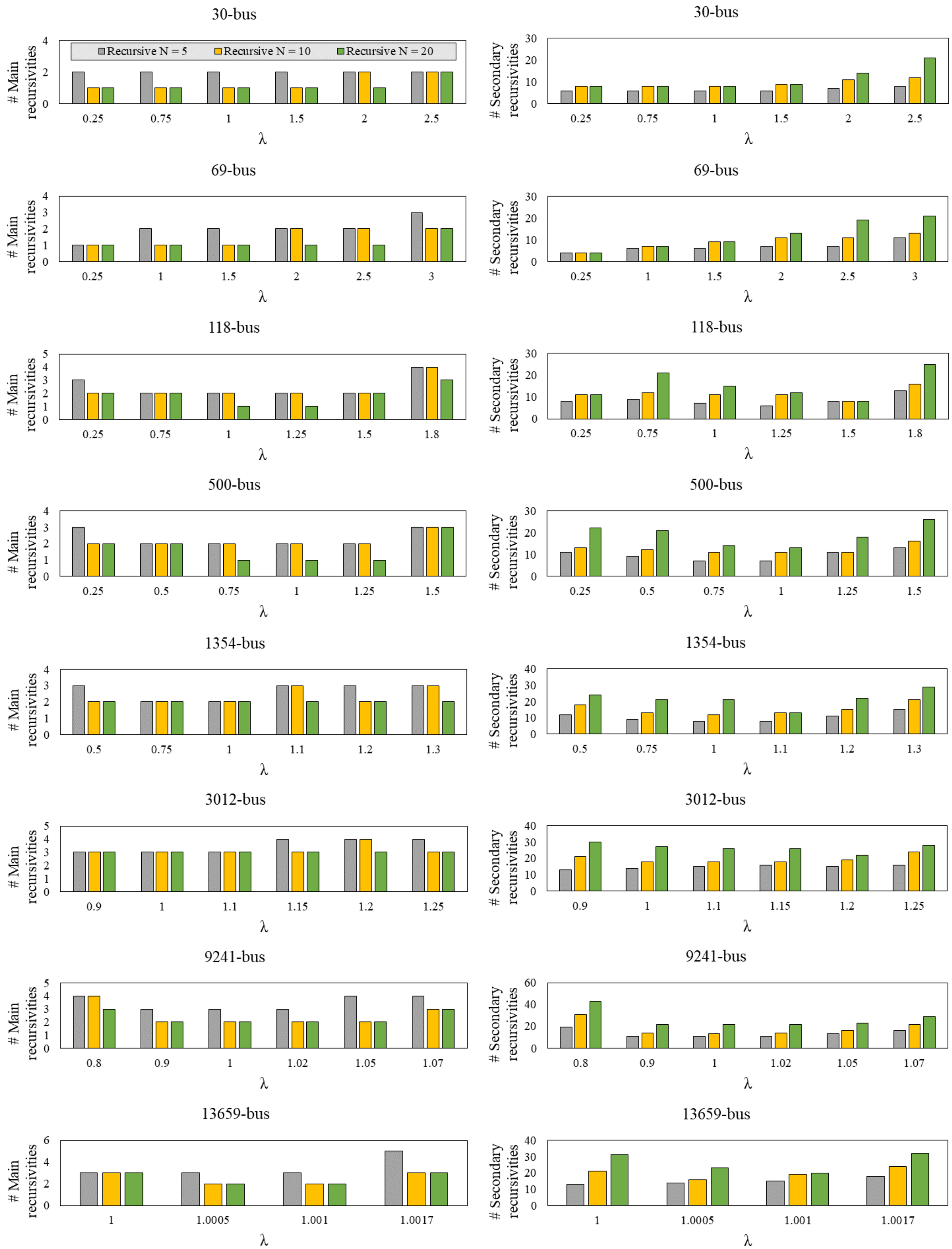

Fig. 7 Total number of main and secondary recursivities for different loading levels

TABLE II
CONVERGENCE RATES WITH REACTIVE LIMITS

	NR	OMP	Recursive – Main (Secondary) recursivities		
			$N = 5$	$N = 10$	$N = 20$
30-bus	6	178	3 (6)	2 (8)	2 (8)
69-bus	3	62	2 (6)	1 (7)	1 (7)
118-bus	6	140	3 (7)	3 (11)	2 (15)
500-bus	9	42	4 (7)	4 (11)	3 (13)
1354-bus	10	666	5 (8)	4 (12)	4 (21)
3012-bus	Fail	940	5 (14)	5 (18)	5 (27)
9241-bus	13	990	7 (11)	5 (13)	5 (22)
13659-bus	Fail	97*	4 (13)	4 (21)	4 (31)

** Low Voltage Solution*

D. Influence of the Loading Level

In this subsection, we study how the loading level affects the overall performance of proposed PF methodology. To do that, the active and reactive powers injected at PQ buses, along the active power injected at PV buses, are multiplied by a factor $\lambda \in \mathbb{R}^+$. Fig. 7 shows the total number of main and secondary recursivities required by the proposed PF methodology for different loading levels. As preceding simulations, results have been obtained from a flat start with $\varepsilon = 10^{-6}$. As observed, as the loading level grows, the convergence rate is affected and, typically, more recursivites are needed for convergence. As seen in this figure, less main recursivities are normally computed for higher values of N. However, the behaviour is totally the opposite regarding the secondary recursivites since, as expected, more recursivities are needed for higher N. Here, obtaining an optimal balance between the total number of main and secondary recursivities is the key idea. In this case, it is expected more efficiency and, consequently, more competitive execution times as less main recursivities are computed since them bring the heaviest calculations. Nevertheless, too many secondary recursivities may lead to numerical stability problems. In this situation, the sanitary step (13) is activated and no more secondary recursivities are computed. So that, the Algorithm 1 works similarly for values $N > 20$.

VI. CONCLUDING REMARKS AND FUTURE WORKS

This paper has proposed a novel methodology for solving PF problems. This methodology is based on recursive formula instead of the conventional iterative methods. Numerical stability of the proposed methodology has been proved. In addition, some important computational aspects and potential failures have been commented.

The proposed approach has been tested in a wide variety of well and ill-conditioned cases. Its performance has been compared with other conventional PF solution methods. The results obtained were promising. The developed recursion function is computationally competitive, scalable and reliable.

Future works should be focused on developing more sophisticated recursive functions, maybe based on some modern robust PF iterative solution techniques like that developed in [13].

REFERENCES

[1] L. Powell, *Power System Load Flow Analysis*. New York: McGraw-Hill, 2005.

[2] F. Milano, "Continuous Newton's Method for Power Flow Analysis," *IEEE Trans. Power Syst.*, vol. 24, no. 1, pp. 50-57, Feb. 2009.

[3] J. B. Ward and H. W. Hale, "Digital computer solution of power-flow problems," *Trans. Amer. Inst. Elect. Eng. Part III, Power App. Syst.*, vol. 75, no. 3, pp. 398-404, Jan. 1956.

[4] W. F. Tinney and C. E. Hart, "Power Flow Solution by Newton's Method," *IEEE Trans. Power Appar. Syst.*, vol. PAS-86, no. 11, pp. 1449-1460, Nov. 1967.

[5] B. Stott and O. Alsac, "Fast Decoupled Load Flow," *IEEE Trans. Power Appar. Syst.*, vol. PAS-93, no. 3, pp. 859-869, May 1974.

[6] M. S. Sachdev and T. K. P. Medicherla, "A second order load flow technique," *IEEE Trans. Power Appar. Syst.*, vol. 96, no. 1, pp. 189-197, Jan. 1977.

[7] S. Iwamoto and Y. Tamura, "A Load Flow Calculation Method for Ill-Conditioned Power Systems," *IEEE Trans. Power Appar. Syst.*, vol. PAS-100, no. 4, pp. 1736-1743, April 1981.

[8] B. Stott, "Review of load-flow calculation methods," *Proc. IEEE*, vol. 62, no. 7, pp. 916-929, July 1974.

[9] W. H. Kersting, *Distribution System Modeling and Analysis*, 2nd ed. Boca Raton, FL: CRC Press, 2006.

[10] F. Milano, *Power System Modelling & Scripting*. New York: Springer, 2010.

[11] S. Y. Derakhshandeh and R. Pourbagher, "Application of high-order Newton-like methods to solve power flow equations," *IET Gener. Transmiss. Distrib.*, vol. 10, no. 8, pp. 1853-1859, May 2016.

[12] F. Milano, "Analogy and Convergence of Levenberg's and Lyapunov-Based Methods for Power Flow Analysis," *IEEE Trans. Power Syst.*, vol. 31, no. 2, pp. 1663-1664, March 2016.

[13] M. Tostado-Véliz, S. Kamel and F. Jurado, "A Robust Power Flow Algorithm Based on Bulirsch-Stoer Method," *IEEE Trans. Power Syst.*, vol. 34, no. 4, pp. 3081-3089, July 2019.

[14] F. Milano, "Implicit Continuous Newton Method for Power Flow Analysis," *IEEE Trans. Power Syst.*, vol. 34, no. 4, pp. 3309-3311, July 2019.

[15] M. Tostado-Véliz, S. Kamel and F. Jurado, "Promising framework based on multistep continuous Newton scheme for developing robust PF methods," *IET Gener. Transmiss. Distrib.*, vol. 14, no. 2, pp. 265-274, Jan. 2020.

[16] A. Trias and J. L. Marín, "The Holomorphic Embedding Loadflow Method for DC Power Systems and Nonlinear DC Circuits," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 63, no. 2, pp. 322-333, Feb. 2016.

[17] S. Rao, Y. Feng, D. J. Tylavsky and M. K. Subramanian, "The Holomorphic Embedding Method Applied to the Power-Flow Problem," *IEEE Trans. Power Syst.*, vol. 31, no. 5, pp. 3816-3828, Sept. 2016.

[18] S. Rao, D.J. Tylavsky and Y. Feng, "Estimating the saddle-node bifurcation point of static power systems using the holomorphic embedding method," *Int. J. Elect. Power Energy Syst.*, vol. 84, pp. 1-12, Jan. 2017.

[19] Y. Feng and D. J. Tylavsky, "A Holomorphic embedding approach for finding the Type-1 power-flow solutions," *Int. J. elect. Power Energy Syst.*, vol. 102, pp. 179-188, Nov. 2018.

[20] M. Karimi, A. Shahriari, M. R. Aghamohammadi, H. Marzooghi and V. Terzija, "Application of Newton-based load flow methods for determining steady- state condition of well and ill-conditioned power systems: A review," *Int. J. Elect. Power Energy Syst.*, vol. 113, pp. 298-309, Dec. 2019.

[21] S. Y. Derakhshandeh and R. Pourbagher, "Application of high-order Newton-like methods to solve power flow equations," *IET Gener. Transmiss. Distrib.*, vol. 10, no. 8, pp. 1853-1859, May 2016.

[22] M. Tostado-Véliz, S. Kamel and F. Jurado, "A novel family of efficient Power-Flow methods with high convergence rate suitable for large realistic power systems," *IEEE Syst. Journal*., to be published.

[23] M. Tostado-Véliz, S. Kamel and F. Jurado, "Robust and efficient approach based on Richardson extrapolation for solving badly initialised/ill-conditioned power-flow problems," *IET Gener. Transmiss. Distrib.*, vol. 13, no. 16, pp. 3524-3533, Aug. 2019.

[24] J.A. Ezquerro and M.A. Hernández, "An optimization of Chebyshev's method," *J. Complex.*, vol. 25, no. 4, pp. 343-361, Aug. 2009.

[25] J. E. Tate and T. J. Overbye, "A comparison of the optimal multiplier in polar and rectangular coordinates," *IEEE Trans. Power Syst.*, vol. 20, no. 4, pp. 1667-1674, Nov. 2005.
[26] S. Kamel, M. Abdel-Akher and F. Jurado, "Improved NR current injection load flow using power mismatch representation of PV bus," *Int. J. Elect. Power Energy Syst.*, vol. 53, pp. 64-68, Dec. 2013.
[27] S. A. Saleh, "The Formulation of a Power Flow Using *d-q* Reference Frame Components-Part I: Balanced 3φ Systems," *IEEE Trans. Ind. Appl.*, vol. 52, no. 5, pp. 3682-3693, Sept.-Oct. 2016.
[28] B. Sereeter, C. Vuik and C. Witteveen, "On a comparison of Newton-Raphson solvers for power flow problems," *J. Comput. Appl. Math.*, vol. 360, pp. 157-169, Nov. 2019.
[29] R. Pires, L. Mili and G. Chagas, "Robust complex-valued Levenberg-Marquardt algorithm as applied to power flow analysis," *Int. J. Elect. Power Energy Syst.*, vol. 113, pp. 383-392, Dec. 2019.
[30] P. A. N. Garcia, J. L. R. Pereira, S. Carneiro, M. P. Vinagre and F. V. Gomes, "Improvements in the representation of PV buses on three-phase distribution power flow," *IEEE Trans. Power Delivery*, vol. 19, no. 2, pp. 894-896, April 2004.
[31] A. Gómez-Expósito, E. Romero-Ramos and I. Dzafic, "Hybrid real-complex current injection-based load flow formulation," *Elect. Power Syst. Res.*, vol. 119, pp. 237-246, Feb. 2015.
[32] A. B. Birchfield, T. Xu, K. M. Gegner, K. S. Shetye and T. J. Overbye, "Grid Structural Characteristics as Validation Criteria for Synthetic Networks," *IEEE Trans. Power Syst.*, vol. 32, no. 4, pp. 3258-3265, Jul. 2017.
[33] J. Lee and H.-D. Chiang, "Convergent regions of the Newton homotopy method for nonlinear systems: theory and computational applications," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 48, no. 1, pp. 51-66, Jan. 2001.
[34] Available within Matpower benchmark systems. [Online]. Available: https://matpower.org/download/ [Accessed 13 Mar. 2020].
[35] C. Josz, S. Fliscounakis, J. Maeght, and P. Panciatici, "AC Power Flow Data in Matpower and QCQP Format: iTesla, RTE Snapshots, and PEGASE". *arXiv: 1603.01533*. [Online] Available: https://arxiv.org/abs/1603.01533 [Accessed 14 Mar. 2020]
[36] S. Fliscounakis, P. Panciatici, F. Capitanescu and L. Wehenkel, "Contingency Ranking With Respect to Overloads in Very Large Power Systems Taking Into Account Uncertainty, Preventive, and Corrective Actions," *IEEE Trans. Power Syst.*, vol. 28, no. 4, pp. 4909-4917, Nov. 2013.
[37] R. D. Zimmerman, C. E. Murillo-Sánchez and R. J. Thomas, "MATPOWER: Steady-State Operations, Planning, and Analysis Tools for Power Systems Research and Education," *IEEE Trans. Power Syst.*, vol. 26, no. 1, pp. 12-19, Feb. 2011.
[38] L. M. C. Braz, C. A. Castro and C. A. F. Murati, "A critical evaluation of step size optimization based load flow methods," *IEEE Trans. Power Syst.*, vol. 15, no. 1, pp. 202-207, Feb. 2000.

Marcos Tostado-Véliz was born in Spain in 1987. He received the Electrical Engineering and Master's degrees in 2016 and 2017, respectively, from the University of Seville, Seville, Spain. He is currently working toward the Ph.D. degree in University of Jaén, Jaén, Spain. His primary research interests include optimal power system planning, operation, and control and numerical methods for power system analysis.

Salah Kamel received the international PhD degree from University of Jaen, Spain (Main) and Aalborg University, Denmark (Host). From Nov. 2014, he joined the Electrical Engineering Department, Faculty of Engineering, Aswan University, Egypt as an Assistant Professor. He is currently a Postdoctoral Research Fellow in State Key Laboratory of Power Transmission Equipment and System Security and New Technology, School of Electrical Engineering, Chongqing University, Chongqing, China. His research activities include power system modeling, analysis and optimization, renewable energy and smart grid technologies.

Francisco Jurado (M'00–SM'06) was born in Linares, Jaén, Spain. He received the M.Sc. and Dr. Ing. degrees from the National University of Distance Education, Madrid, Spain, in 1995 and 1999, respectively. Since 1985, he has been a Professor with the Department of Electrical Engineering, University of Jaén, Jaén. His current research interests include power systems, modeling, and renewable energy.