



Decision Support

A Cost Consensus Metric for Consensus Reaching Processes based on a comprehensive minimum cost model

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ABSTRACT

Consensus Reaching Processes (CRPs) have recently acquired much more importance within Group Decision Making real-world problems because of the demand of either agreed or consensual solutions in such decision problems. Hence, many CRP models have been proposed in the specialized literature, but so far there is not any clear objective to evaluate their performance in order to choose the best CRP model. Therefore, this research aims at developing an objective metric based on the cost of modifying experts' opinions to evaluate CRPs in GDM problems. First, a new and comprehensive minimum cost consensus model that considers distance to global opinion and consensus degree is presented. This model obtains an optimal agreed solution with minimum cost but this solution is not dependent on experts' opinion evolution. Therefore, this optimal solution will be used to evaluate CRPs in which experts' opinion evolution is considered to achieve an agreed solution for the GDM. Eventually, a comparative performance analysis of different CRPs on a GDM problem will be provided to show the utility and validity of this cost metric.

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1. Introduction

Decision making is a common process in human being's daily life, characterized by the existence of at least two alternatives and the need of selecting which one is the best solution of the problem. Nowadays, several experts with different points of view often take part in a decision problem with the aim of obtaining a common solution, leading to a Group Decision Making (GDM) problem. Traditionally, the GDM problems have been solved by a selection process (Herrera, Herrera-Viedma, & Verdegay, 1995), but such a process ignores the agreement among experts, which implies that some experts may think that their opinions have not been considered sufficiently. Disagreement among experts is inevitable in most of real world problems, hence it is important to remove the disagreement among experts to obtain an agreed solution that is generally more appreciated by the group and stakeholders as well as demanded by many real world problems. Thus, a Consensus Reaching Process (CRP) has been added in the resolution of GDM problems. In a CRP, experts discuss and modify their preferences to make them closer to each other with the aim of increasing the

level of agreement among experts to obtain an acceptable solution for all of them (Butler & Rothstein, 2007; Dong & Xu, 2016; Herrera-Viedma, Cabrerizo, Kacprzyk, & Pedrycz, 2014; Palomares & Martínez, 2014; Palomares, Martínez, & Herrera, 2014b; Palomares, Rodríguez, & Martínez, 2013; Xu, Du, & Chen, 2015). There are different interpretations of consensus, from unanimous agreement among the group to more flexible soft consensus (Cabrerizo, Moreno, Pérez, & Herrera-Viedma, 2010; Kacprzyk & Fedrizzi, 1988; Kacprzyk, Nurmi, & Fedrizzi, 1997; Kacprzyk & Zadrożny, 2010; Kacprzyk, Zadrożny, & Raś, 2010; Zhang, Kou, & Peng, 2019). In the literature there are many consensus models (Herrera-Viedma et al., 2014; Palomares, Estrella, Martínez, & Herrera, 2014a; Zhang, Dong, Chiclana, & Yu, 2019). Palomares et al. (2014a) provided a comprehensive taxonomy of CRPs based on two dimensions (see Fig. 1):

- Consensus with feedback and without feedback.
- Consensus measures based on distances to the collective opinions and based on distances between experts.

In the horizontal axis, the CRPs 'without feedback' achieve consensus by modifying initial opinions without consider experts meanwhile, CRPs 'with feedback', involve discussions among experts and they should modify their opinions to reach a consensus.

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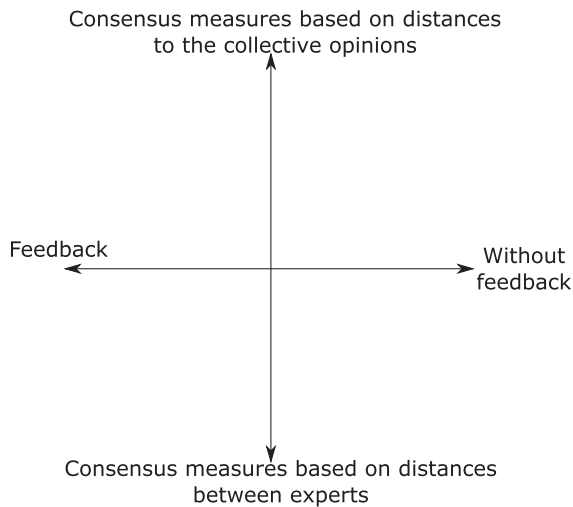


Fig. 1. A taxonomy for consensus approaches.

Despite there are many proposals on CRPs (Cheng, Zhou, Cheng, Zhou, & Xie, 2018; Chiclana, Mata, Martínez, Herrera-Viedma, & Alonso, 2008; Herrera-Viedma, Herrera, & Chiclana, 2002; Kacprzyk & Zadrozny, 2010; Rodríguez, Labella, De Tré, & Martínez, 2018; Wu, Kou, & Peng, 2018), there is no any suitable criteria to evaluate and compare the CRPs, and show which one has a better performance for a given GDM problem. Recently, the software AFRYCA (A FRamework for the aNalysis of Consensus Approaches) (Labella, Liu, Rodríguez, & Martínez, 2018; Palomares et al., 2014a) was a first attempt to facilitate the analysis and comparison among different CRPs' performance. This tool uses several measures such as the number of rounds necessary to reach consensus, number of changes carried out across the CRP and different consistency measures as criteria to compare the performance of different CRP models (Labella et al., 2018). However, these criteria are quite simple and cannot objectively and adequately measure the performance of different CRP models. Thus, our objective is to define an objective metric to evaluate the performance of a CRP taking into account the cost of modifying experts' opinion.

Several researchers have pointed out the importance of considering cost of modifying experts' opinion to reach consensus and it has become an attractive challenge to tackle in CRPs. Ben-Arieh and Easton (2007) defined the concept of minimum-cost consensus (MCC) and proposed the first MCC model which uses a linear cost function to achieve consensus. Afterwards, Ben-Arieh, Easton, and Evans (2008) introduced another MCC model by using a quadratic cost function, and taking into account these two models as a base, some new MCC models have been proposed (Gong, Zhang, Forrest, Li, & Xu, 2015; Li, Zhang, & Dong, 2017; Liu, Chan, Li, Zhang, & Deng, 2012; Zhang, Dong, & Xu, 2013; Zhang, Dong, Xu, & Li, 2011; Zhang, Gong, & Chiclana, 2017). Nevertheless, all these models, modify experts' opinions automatically without experts' supervision and consider just a consensus measure, the distance of each expert to the collective opinion, ignoring a minimum agreement among all experts. It must be highlighted that small distances among experts and the collective opinion cannot always ensure a required acceptable consensus level. Therefore, it is necessary to consider a more comprehensive cost model that not only uses the distance of each expert to the collective opinion, but also reaches a minimum agreement among experts to obtain better and more acceptable solutions.

In this paper, first several novel MCC models that integrate both consensus measures, (i) distance to collective opinion and (ii) minimum agreement, are introduced. Such MCC models will achieve

optimal solutions from each point of view to achieve consensus, but they will not take into account experts for modifying their opinions. Therefore, such an optimal solution will be used to define a Cost Consensus Metric (CCM) that studies the cost performance of CRPs that consider the modification of the experts' opinions to achieve consensus. This CCM will be implemented in the software AFRYCA and a comparative analysis of the cost performance among several CRPs will be carried out to show the results obtained by this new metric.

The remainder of this paper is structured as follows. Section 2 reviews some basic concepts about GDM, CRPs and MCC models. Section 3 presents some new MCC models that consider the distance of each expert to the collective opinion and a minimum agreement among experts to achieve consensus. Section 4 introduces a CCM to evaluate the performance of CRPs. Section 5 provides a comparison experiment of several existing CRP models and analyzes the results by means of AFRYCA. Finally, Section 6 points out some conclusions.

2. Preliminaries

This section makes a short review about basic concepts of GDM, CRP and MCC models, that are necessary to understand our proposal.

2.1. Group Decision Making

GDM problems are very common activities in human's life which consist of a set of experts $E = \{e_1, \dots, e_m\}$, who provide their preferences over a set of possible alternatives or options $X = \{x_1, \dots, x_n\}$, with the aim of obtaining a common solution (Lu, Zhang, Ruan, & Wu, 2007). Each expert $e_k \in E$ expresses his/her opinions over the different alternatives in an information domain by means of a preference structure. There are different preference structures for GDM problems:

- Preference relation: in a preference relation P^k , the assessment p_{ij}^k provided by the expert e_k , represents the preference degree of the alternative x_i over the alternative x_j , $i, j \in \{1, \dots, n\}$. It is shown as follows:

$$P^k = \begin{pmatrix} p_{11}^k & \dots & p_{1n}^k \\ \vdots & \ddots & \vdots \\ p_{n1}^k & \dots & p_{nn}^k \end{pmatrix}.$$

- Decision matrix: in a decision matrix, the assessment p_{ij}^k represents the expert e_k 's opinion over the alternative x_i based on a certain decision criterion c_j , unlike a preference relation that establishes pairwise comparisons between alternatives. It is expressed as follows:

	c^1	c^2	$\dots$	c^l
x_1	p_{11}	p_{12}	$\dots$	p_{1l}
x_2	p_{21}	p_{22}	$\dots$	p_{2l}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	p_{n1}	p_{n2}	$\vdots$	p_{nl}

There are different preference relations depending on the expression domain, such as fuzzy preference relation (FPR), linguistic preference relation, hesitant preference relation (De Baets & Fodor, 1997; Rodríguez, Xu, Martínez, & Herrera, 2018), etc. The use of FPR facilitates the preference elicitation to experts by means of pairwise comparison in a continuous scale in $[0,1]$, due to its simplicity and easy construction it is one of the most widely-used preferences structures in GDM to elicit experts' preferences. Even

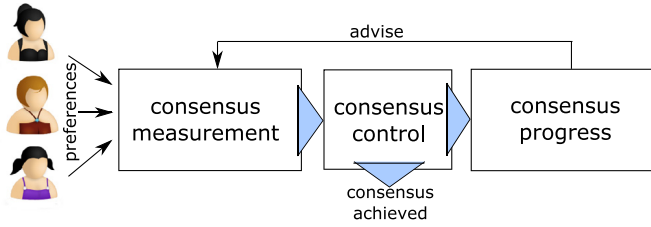


Fig. 2. General scheme of a CRP.

though in the future our research proposals can be studied in other type of preference relations.

Definition 1. Orlovsky (1978) A fuzzy preference relation P^k , associated to an expert e_k on a set of alternatives X , is a fuzzy set on $X \times X$, characterized by the membership function $\mu_{P^k} : X \times X \rightarrow [0, 1]$. When the number of alternatives n is finite, P^k is represented by a $n \times n$ matrix of assessments $\mu_{P^k}(x_i, x_j) = p_{ij}^k$ as follows:

$$P^k = \begin{pmatrix} p_{11}^k & \cdots & p_{1n}^k \\ \vdots & \ddots & \vdots \\ p_{n1}^k & \cdots & p_{nn}^k \end{pmatrix},$$

where each assessment p_{ij}^k represents the *degree of preference* of the alternative x_i over x_j according to expert e_k . The fuzzy preference relation is usually assumed to be additive reciprocal, i.e., $p_{ij}^k + p_{ji}^k = 1, \forall i, j = 1, 2, \dots, n, k = 1, 2, \dots, m$.

The classical selection process to solve a GDM problem is divided into two phases:

- **Aggregation:** in this phase, the preference relations provided by experts are fused by means of an aggregation operator to obtain a collective opinion.
- **Exploitation:** it selects the best alternative(s) as solution of the GDM problem by using the result obtained in the previous phase.

Nevertheless, this process does not always guarantee that the decision selected is accepted by all experts involved in the problem, because some of them might consider that their preferences are not taken into account. A common solution to obtain decisions accepted by the whole group of experts, is to remove the disagreement among them. To do so, a CRP is incorporated before the selection process Saint and Lawson (1994).

2.2. Consensus Reaching Process and consensus measures

A CRP is an iterative and dynamic process in which experts discuss and modify their initial preferences with the aim of achieving a collective opinion that satisfies all experts involved in the GDM problem. Such a process is usually guided and supervised by a human figure known as *moderator*. There are many consensus models (Chiclana et al., 2008; Dong, Zhu, & Cooper, 2017; Herrera-Viedma et al., 2002; Kacprzyk & Zadrożny, 2010; Rodríguez et al., 2018), in Palomares et al. (2014a) a taxonomy and a deep revision about some of them were proposed. A general scheme of a CRP is sketched in Fig. 2 and briefly described as follows:

- **Consensus measurement:** the preferences provided by experts are gathered, and the level of agreement in the group is computed by means of consensus measures (Beliakov, Calvo, & James, 2014) which are based on distance measures and aggregation operators (Montserrat-Adell, Agell, Sánchez, & Ruiz, 2018).

- **Consensus control:** the level of agreement computed is compared with the consensus threshold, $\alpha \in [0, 1]$, fixed a priori. If the level of agreement is greater than the consensus threshold, a selection process is applied, otherwise it is necessary to carry out another discussion round. To avoid an excessive number of rounds, a parameter that indicates the maximum number of rounds allowed, $Maxround \in \mathbb{N}$, is considered.
- **Consensus progress:** the moderator identifies the experts' preferences causing disagreement and advises them to modify such preferences to increase the level of agreement in the next round.

A key phase in the scheme is the first one, *Consensus measurement*, that computes the current level of agreement in the group. According to the taxonomy introduced in Palomares et al. (2014a), the consensus measures can be classified in two types Beliakov et al. (2014):

- Consensus measure based on the distance of each expert to the collective opinion given by the following equation:

$$consensus(o_1, \dots, o_m) = 1 - f_2(f_1(d(o_i, o^c))), \quad (1)$$

where $(o_1, \dots, o_m)$ are the assessments provided by experts $(e_1, \dots, e_m)$ over an alternative, o^c is the collective opinion, $d(\cdot, \cdot)$ is a distance measure, and $f_1 : R^+ \rightarrow R^+, f_2 : R^+ \rightarrow [0, 1]$ are functions.

- Consensus measure based on the distances among experts given by the following formula:

$$consensus(o_1, \dots, o_m) = 1 - g_2(g_1(d(o_i, o_j))), i \neq j, \quad (2)$$

where $g_1 : R^+ \rightarrow R^+, g_2 : R^+ \rightarrow [0, 1]$ are functions, and other symbols are the same as in Eq. (1).

2.3. Minimum-cost consensus models

In CRPs the cost of modifying experts' preferences is key in the collective opinion. Ben-Arieh and Easton (2007) introduced the concept of *minimum-cost consensus* and proposed a MCC model that defines the consensus as the minimum distance between each expert and the collective opinion. This model seeks to minimize the overall cost of moving all experts' opinions by using a linear function.

Definition 2. Zhang et al. (2011) Let $(o_1, \dots, o_m)$ be the original assessments provided by a set of experts $E = \{e_1, e_2, \dots, e_m\}$ over an alternative. Suppose that after CRP, the experts' assessments are modified into $(\bar{o}_1, \dots, \bar{o}_m)$, and a collective opinion $\bar{o}$ is obtained based on the modified assessments, and $(c_1, \dots, c_m)$ are the cost of moving each expert's opinion 1 unit, respectively. The parameter ε is the maximum acceptable distance of each expert to the collective opinion. The MCC model based on a linear cost function is given as follows:

(M-1)

$$\min \sum_{k=1}^m c_k |\bar{o}_k - \bar{o}|$$

s.t. $|\bar{o}_k - \bar{o}| \leq \varepsilon, k = 1, 2, \dots, m$.

It is noteworthy that ε in the model (M-1) measures the absolute deviation between each expert's adjusted opinion and the collective opinion, and it is not necessary to be valued in $[0, 1]$. According to this model, an expert's opinion does not need to be changed if it is in the interval $[\bar{o} - \varepsilon, \bar{o} + \varepsilon]$, and any expert's initial opinion further than ε from $\bar{o}$ should only be changed until that expert's opinion is exactly ε away from $\bar{o}$.

Taking into account (M-1), Zhang et al. (2011) studied how the level of agreement in the group can be different according to the

selected aggregation operator for computing the collective opinion, and proposed a new MCC model as follows:

(M – 2)

$$\min \sum_{i=1}^m c_i |\bar{o}_i - o_i|$$

$$s.t. \begin{cases} \bar{o} = F(\bar{o}_1, \dots, \bar{o}_m) \\ |\bar{o}_i - \bar{o}| \leq \varepsilon, i = 1, 2, \dots, m, \end{cases}$$

where F is an aggregation function.

The properties of (M–2) were investigated under the situations that F can be the weighted average operator or the OWA operator Zhang et al. (2011).

Recently, some researchers have paid much attention on the model proposed by Ben-Arieh and Easton (2007) and have introduced some new MCC approaches (Gong et al., 2015; Li et al., 2017; Liu et al., 2012; Zhang et al., 2013; Zhang et al., 2017). However, all these models present a disadvantage, that is, they only consider the distance of each expert to the collective opinion ignoring a minimum agreement among experts to reach consensus that is the main measure considered in many CRPs (Chiclana et al., 2008; Kacprzyk & Zadrozny, 2010; Wu & Xu, 2016; Zhang, Dong, & Xu, 2012). Therefore, the overall opinion obtained cannot guarantee a required consensus degree for all experts involved in the GDM problem and a comprehensive MCC should be developed.

3. New MCC models considering the distance and consensus degree

This section proposes several new MCC models which cope with the previous drawback of the existing MCC models. Therefore, with the aim of defining a comprehensive MCC model that takes into account level of agreement and distance to collective opinion, first a MCC model that deals with single numerical values is defined and it is then extended to deal with FPRs.

3.1. MCC models dealing with numerical values

As it was aforementioned, small distances between experts and the collective opinion cannot always ensure that experts reach a high consensus level. Therefore, it is necessary to define a new MCC model that is able to achieve a minimum agreement among experts to obtain better consensual solutions. Thus, the model (M–2) is modified including the computation of consensus level. The model obtained is the following one:

(M – 3)

$$\min \sum_{i=1}^m c_i |\bar{o}_i - o_i|$$

$$s.t. \begin{cases} \bar{o} = F(\bar{o}_1, \dots, \bar{o}_m) \\ |\bar{o}_i - \bar{o}| \leq \varepsilon, i = 1, 2, \dots, m \\ \text{consensus}(\bar{o}_1, \dots, \bar{o}_m) \geq \alpha, \end{cases}$$

where $\text{consensus}(\cdot)$ represents the consensus level achieved, $\alpha \in [0, 1]$ is a consensus threshold fixed a priori, F is an aggregation operator, and ε is a parameter measures distance between each expert's adjusted opinion and the collective opinion.

Remark 1. Taking into account that the condition $\text{consensus}(\bar{o}_1, \dots, \bar{o}_m) \geq \alpha$, Eqs. (1) and (2) related to the consensus measures can be transformed into the following inequalities:

$$f_2(f_1(d(o_i, o^c))) \leq 1 - \alpha, \quad (3)$$

or

$$g_2(g_1(d(o_i, o_j))) \leq 1 - \alpha, i \neq j. \quad (4)$$

Since there are two ways of computing consensus, two MCC models can be defined according to the consensus measures.

- **Consensus measure based on the distance between experts and the collective opinion.** In this case, the distance can be measured by $|\bar{o}_i - \bar{o}|, i = 1, \dots, m$. Experts might have different importance in the consensus process. Therefore, without loss of generality, the operator to aggregate the distances could be the Weighted Average operator, that is, $\sum_{i=1}^m w_i |\bar{o}_i - \bar{o}|$, where $w_i \in [0, 1]$ is the expert e_i 's weight and $\sum_{i=1}^m w_i = 1$. Therefore, the model (M – 3) can be transformed into the following one:

(M – 4)

$$\min \sum_{i=1}^m c_i |\bar{o}_i - o_i|$$

$$s.t. \begin{cases} \bar{o} = \sum_{i=1}^m w_i \bar{o}_i \\ |\bar{o}_i - \bar{o}| \leq \varepsilon, i = 1, 2, \dots, m \\ \sum_{i=1}^m w_i |\bar{o}_i - \bar{o}| \leq \gamma, \end{cases}$$

where $\gamma = 1 - \alpha, \varepsilon \in [0, 1]$ measures the deviation between each expert's adjusted opinion and the collective opinion.

In order to justify the consensus measure used in (M–4), we use the similar consensus measure proposed in Chiclana et al. (2008), then

$$\text{consensus}(\bar{o}_1, \dots, \bar{o}_m) = 1 - \frac{1}{m} |\bar{o}_i - \bar{o}|.$$

Thus the condition

$$\text{consensus}(\bar{o}_1, \dots, \bar{o}_m) \geq \alpha$$

can be transformed into

$$\frac{1}{m} |\bar{o}_i - \bar{o}| \leq 1 - \alpha = \gamma.$$

This approach actually adopts an assumption that each expert has the same contribution to overall consensus. However, we think that the model should offer the view that important experts should contribute more to the consensus. Suppose that in a GDM problem, the expert e_1 has an importance weight $w_1 = 0.9$. It is then reasonable to think that the consensus is almost acceptable if e_1 's adjusted opinion $\bar{o}_1$ is close enough to the collective opinion $\bar{o}$. Therefore, we improve the consensus model as follows:

$$\text{consensus}(\bar{o}_1, \dots, \bar{o}_m) = \sum_{i=1}^m w_i |\bar{o}_i - \bar{o}|.$$

Accordingly, the requirement

$$\text{consensus}(\bar{o}_1, \dots, \bar{o}_m) \geq \alpha$$

is transformed into

$$\sum_{i=1}^m w_i |\bar{o}_i - \bar{o}| \leq 1 - \alpha = \gamma.$$

Remark 2. Note that model (M–4) has been defined by considering that the original values $(o_1, o_2, \dots, o_m)$ are assessed in $[0, 1]$. Appendix A shows the transformation of the model (M–4) with non-normalized values.

- **Consensus measure based on the distance among experts.** Given an expert e_i , the distances between e_i and the remaining experts e_j is computed by $|\bar{o}_i - \bar{o}_j|, \forall j = 1, \dots, m, j \neq i$, and the average distance among them is obtained as follows:

$$\frac{1}{m-1} \sum_{i=0, j \neq i}^m |\bar{o}_i - \bar{o}_j|, i = 1, \dots, m. \quad (5)$$

Table 1
The costs with different values of ε and γ of (M-4).

	$\gamma = 0.3$	$\gamma = 0.25$	$\gamma = 0.2$	$\gamma = 0.15$	$\gamma = 0.1$	$\gamma = 0.05$
$\varepsilon = 0.3$	1.01	1.01	1.01	1.55	2.26	3
$\varepsilon = 0.25$	1.13	1.13	1.13	1.55	2.26	3
$\varepsilon = 0.2$	1.32	1.32	1.32	1.55	2.26	3
$\varepsilon = 0.15$	1.81	1.81	1.81	1.81	2.26	3
$\varepsilon = 0.1$	2.47	2.47	2.47	2.47	2.47	3
$\varepsilon = 0.05$	3.13	3.13	3.13	3.13	3.13	3.13

Considering that experts might have different importance in the CRP and without loss of generality, the distances can be aggregated by means of the Weighted Average operator as follows:

$$\begin{aligned} & \frac{w_1}{m-1} \sum_{j \neq 1} |\bar{o}_1 - \bar{o}_j| + \frac{w_2}{m-1} \sum_{j \neq 2} |\bar{o}_2 - \bar{o}_j| + \dots + \frac{w_m}{m-1} \sum_{j \neq m} |\bar{o}_m - \bar{o}_j| \\ &= \sum_{i=1}^{m-1} \sum_{j=i+1}^m \frac{w_i + w_j}{m-1} |\bar{o}_i - \bar{o}_j|. \end{aligned} \quad (6)$$

Therefore, the consensus level can be obtained as follows:

$$\text{consensus}(\bar{o}_1, \dots, \bar{o}_n) = \sum_{i=1}^{m-1} \sum_{j=i+1}^m \frac{w_i + w_j}{m-1} |\bar{o}_i - \bar{o}_j|. \quad (7)$$

By using Eq. (7), the model (M-3) can be transformed into the following one:

(M-5)

$$\begin{aligned} & \min \sum_{i=1}^m c_i |\bar{o}_i - o_i| \\ & \text{s.t.} \begin{cases} \bar{o} = \sum_{i=1}^m w_i \bar{o}_i \\ |\bar{o}_i - \bar{o}| \leq \varepsilon, i = 1, 2, \dots, m \\ \sum_{i=1}^{m-1} \sum_{j=i+1}^m \frac{w_i + w_j}{m-1} |\bar{o}_i - \bar{o}_j| \leq \gamma, \end{cases} \end{aligned}$$

where ε and γ are the same as in the model (M-4).

Remark 3. Note that model (M-5) has been defined by considering that the original values $(o_1, o_2, \dots, o_m)$ are assessed in $[0, 1]$. Appendix A shows the transformation of the model (M-5) with non-normalized values.

In order to show the performance of the proposed models, we present a numerical example.

Example 1. Let us consider a numerical example in which there are five experts $E = \{e_1, \dots, e_5\}$ with weights $W = (0.375, 0.1875, 0.25, 0.0625, 0.125)$, who provide their assessments over an alternative as $(o_1, \dots, o_5) = (0, 0.09, 0.36, 0.45, 1)$. The costs are $(c_1, \dots, c_5) = (6, 3, 4, 1, 2)$, where c_k is the cost of modifying the opinion of the expert e_k . By using the model (M-4), the minimum costs regarding several different values of ε and γ are shown in Table 1. Note that, a high consensus threshold means a small value of γ , therefore the values used for γ are less than 0.30 to show the behaviour of the model (M-4).

By using the model (M-4), Table 2 shows in further detail the optimal solutions and minimum cost with $\varepsilon = 0.2$, $\gamma = 0.05, 0.1, 0.15, 0.2, 0.25, 0.3$ and Table 3 shows the optimal solutions and minimum cost with $\varepsilon = 0.145$, $\gamma = 0.05, 0.1, 0.15, 0.2, 0.25, 0.3$.

Analysis of results

From Table 1, we can see that for a fixed ε , at first the minimum cost is constant, then it increases. For example, for $\varepsilon = 0.3$, when $\gamma = 0.3, 0.25, 0.2, 0.15, 0.1, 0.05$, the minimum costs are 1.01, 1.01, 1.01, 1.55, 2.26, 3, respectively. The reason is because for a big

Table 2
The optimal solutions of model (M-4) with $\varepsilon = 0.2$.

γ	$\bar{o}_1$	$\bar{o}_2$	$\bar{o}_3$	$\bar{o}_4$	$\bar{o}_5$	$\bar{o}$	Cost
0.3	0	0.09	0.36	0.38	0.38	0.18	1.32
0.25	0	0.09	0.36	0.38	0.38	0.18	1.32
0.2	0	0.09	0.36	0.38	0.38	0.18	1.32
0.15	0	0.09	0.31	0.36	0.36	0.16	1.55
0.1	0	0.09	0.22	0.12	0.32	0.12	2.26
0.05	0.02	0.09	0.19	0.09	0.09	0.09	3

Table 3
The optimal solutions of model (M-4) with $\varepsilon = 0.145$.

γ	$\bar{o}_1$	$\bar{o}_2$	$\bar{o}_3$	$\bar{o}_4$	$\bar{o}_5$	$\bar{o}$	Cost
0.3	0	0.09	0.29	0.29	0.29	0.14	1.88
0.25	0	0.09	0.29	0.29	0.29	0.14	1.88
0.2	0	0.09	0.29	0.29	0.29	0.14	1.88
0.15	0	0.09	0.29	0.29	0.29	0.14	1.88
0.1	0	0.09	0.25	0.12	0.26	0.12	2.26
0.05	0.02	0.09	0.19	0.09	0.09	0.09	3

Table 4
The costs with different values of ε and γ of (M-5).

	$\gamma = 0.3$	$\gamma = 0.25$	$\gamma = 0.2$	$\gamma = 0.15$	$\gamma = 0.1$	$\gamma = 0.05$
$\varepsilon = 0.30$	1.01	1.20	1.60	2.14	2.69	3.24
$\varepsilon = 0.25$	1.13	1.20	1.60	2.14	2.69	3.24
$\varepsilon = 0.20$	1.32	1.32	1.60	2.14	2.69	3.24
$\varepsilon = 0.15$	1.81	1.81	1.81	2.14	2.69	3.24
$\varepsilon = 0.10$	2.47	2.47	2.47	2.47	2.69	3.24
$\varepsilon = 0.05$	3.13	3.13	3.13	3.13	3.13	3.24

value of γ , the minimum cost is determined by ε , and for a small value of γ , the minimum cost is determined by γ . Furthermore, we can also observe from Tables 2 and 3, that for a same value of γ and different values of ε , the optimal solutions of (M-4) are different even though the minimum costs are identical. For example, when $\varepsilon = 0.2$, $\gamma = 0.10$ in Table 2, the minimum cost is 2.26, and the optimal solution of (M-4) is:

$$(\bar{o}_1, \dots, \bar{o}_5, \bar{o}) = (0, 0.09, 0.22, 0.12, 0.32, 0.12).$$

When $\varepsilon = 0.145$, $\gamma = 0.10$ in Table 3, the minimum cost is also 2.26, but the optimal solution of (M-4) is

$$(\bar{o}_1, \dots, \bar{o}_5, \bar{o}) = (0, 0.09, 0.25, 0.12, 0.26, 0.12).$$

We can also use the model (M-5) to solve this example. The minimum costs with respect to several different values of ε and γ are shown in Table 4.

From Table 4, similar conclusions as Table 1 can be obtained and thus they are omitted here.

From Example 1, we can see that both ε and γ play an important role in the models (M-4) and (M-5). In the following we will provide an algorithm to show several rules to select their values.

Algorithm 1

Step 1. Select the value of $\varepsilon \in [0, 1]$ which reflects the permitted maximum deviation between each expert and the collective opinion.

Step 2. Solve the model (M-2) with F being the weighted average operator, and obtain the optimal solution $(\bar{o}_1, \dots, \bar{o}_n, \bar{o})$. Then calculate γ_0 by using:

$$\gamma_0 = \sum_{i=1}^m w_i |\bar{o}_i - \bar{o}|,$$

or

$$\gamma_0 = \sum_{i=1}^{m-1} \sum_{j=i+1}^m \frac{w_i + w_j}{m-1} |\bar{o}_i - \bar{o}_j|,$$

$\bar{o}_i \in [0, 1], i = 1, 2, \dots, m$.

Step 3. Select an accepted consensus threshold α , and compute $\gamma = 1 - \alpha$. Since $\gamma \geq \gamma_0$, the optimal solutions and minimum costs of the model (M-4) or (M-5) will be the same as the model (M-2). Consequently, we suggest to select the values of γ and α with $\gamma \leq \gamma_0$ and $\alpha \geq 1 - \gamma_0$.

3.2. MCC models dealing with FPRs

One of the preference structures most widely used in GDM and hence, in the corresponding CRPs, is the FPR Orlovsky (1978), therefore, the proposed MCC model (M-3) is modified to deal with FPRs.

Let $P^k = (p_{ij}^k)_{n \times n}$ be a FPR provided by an expert $e_k, k = 1, \dots, m$. In order to achieve a solution accepted by all experts involved in the GDM problem, P^k is adjusted to $\bar{P}^k = (\bar{p}_{ij}^k)_{n \times n}, k = 1, \dots, m$, and the collective FPR of the adjusted FPRs is $\bar{P} = (\bar{p}_{ij})_{n \times n}$.

Depending on the consensus measures to compute consensus level, two different MCC models are introduced.

- Using the consensus measure based on the distance between each expert's opinion and the collective opinion.

(M-6)

$$\begin{aligned} \min & \sum_{k=1}^m \sum_{i=1}^{n-1} \sum_{j=i+1}^n c_k |p_{ij}^k - \bar{p}_{ij}^k| \\ \text{s.t.} & \begin{cases} \bar{p}_{ij} = \sum_{k=1}^m w_k \bar{p}_{ij}^k \\ |\bar{p}_{ij}^k - \bar{p}_{ij}| \leq \varepsilon, k = 1, \dots, m, i = 1, \dots, n-1, j = i+1, \dots, n \\ \frac{2}{n(n-1)} \sum_{k=1}^m \sum_{i=1}^{n-1} \sum_{j=i+1}^n w_k |\bar{p}_{ij}^k - \bar{p}_{ij}| \leq \gamma. \end{cases} \end{aligned}$$

- Using the consensus measure based on the distance among experts.

(M-7)

$$\begin{aligned} \min & \sum_{k=1}^m \sum_{i=1}^{n-1} \sum_{j=i+1}^n c_k |p_{ij}^k - \bar{p}_{ij}^k| \\ \text{s.t.} & \begin{cases} \bar{p}_{ij} = \sum_{k=1}^m w_k \bar{p}_{ij}^k \\ |\bar{p}_{ij}^k - \bar{p}_{ij}| \leq \varepsilon, k = 1, \dots, m, i = 1, \dots, n-1, j = i+1, \dots, n \\ \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^n \sum_{k=1}^{m-1} \sum_{l=k+1}^m \frac{w_k + w_l}{m-1} |\bar{p}_{ij}^k - \bar{p}_{ij}^l| \leq \gamma. \end{cases} \end{aligned}$$

Example 2. Let us consider a numerical example in which there are three experts $E = \{e_1, e_2, e_3\}$ with weights $W = (0.375, 0.250, 0.375)$ and costs $(c_1, c_2, c_3) = (2, 5, 3)$, respectively, who provide their assessments over three alternatives $X = \{x_1, x_2, x_3\}$ by means of FPRs (see Definition 1):

$$P^1 = \begin{pmatrix} 0.5 & 0.87 & 0.99 \\ & 0.5 & 0.91 \\ & & 0.5 \end{pmatrix}, P^2 = \begin{pmatrix} 0.5 & 0.14 & 0.03 \\ & 0.5 & 0.14 \\ & & 0.5 \end{pmatrix},$$

$$P^3 = \begin{pmatrix} 0.5 & 0.43 & 0.02 \\ & 0.5 & 0.03 \\ & & 0.5 \end{pmatrix}.$$

Note that FPRs are assumed to be reciprocal and thus elements in the lower triangular matrix are omitted here.

By applying model (M-6) with $\varepsilon = 0.3$ and $\gamma = 0.2$ the resulting modified preferences are:

$$\bar{P}^1 = \begin{pmatrix} 0.5 & 0.66 & 0.47 \\ & 0.5 & 0.52 \\ & & 0.5 \end{pmatrix}, \bar{P}^2 = \begin{pmatrix} 0.5 & 0.14 & 0.02 \\ & 0.5 & 0.14 \\ & & 0.5 \end{pmatrix},$$

$$\bar{P}^3 = \begin{pmatrix} 0.5 & 0.41 & 0.02 \\ & 0.5 & 0.03 \\ & & 0.5 \end{pmatrix}.$$

The resulting cost of modifying the initial preferences is 2.33.

Thus, the preferences $\bar{P}^1, \bar{P}^2$ and $\bar{P}^3$ represent the experts' preferences with the minimum necessary modifications to achieve the conditions related to the parameters γ and ε and whose total change cost is 2.33.

4. A Cost Consensus Metric based on minimum cost: measuring Consensus Reaching Processes performance

In spite of the multiple CRPs introduced in the specialized literature (Chiclana et al., 2008; Herrera-Viedma et al., 2002; Kacprzyk & Zadrozny, 2010; Rodríguez et al., 2018), new CRP models are commonly introduced but without a clear advantage over the previous ones. So, in order to support the improvement of CRPs, it is necessary to establish an objective and standard measure to analyse the performance of the CRP models. Consequently, our aim is to define a metric that provides a clear and objective measure about the performance of CRPs to justify the development of new CRPs models as the selection of one model among the multiple ones extant in the literature.

4.1. Cost Consensus Metric

Keeping our aim in mind, we propose to measure the cost incurred by the CRP models to reach the consensus as a metric to evaluate their performance in an objective way. Such a metric is so-called Cost Consensus Metric (CCM), and it will consider the optimal solution the one obtained by models (M-6) or (M-7) because it is the minimum possible cost. Therefore, the metric will compute the difference in cost between the MCC model solution and the solution obtained by different evaluated CRPs.

Suppose that the initial experts' preferences are $P = (P^1, \dots, P^m)$ and the optimal adjusted FPRs of the MCC model (M-6) or (M-7) are $\bar{P} = (\bar{P}^1, \dots, \bar{P}^m)$, where P^k and $\bar{P}^k$ are the initial and adjusted FPRs of the expert $e_k, k = 1, 2, \dots, m$, respectively. The distance between P^k and $\bar{P}^k$ can be computed as

$$d(P^k, \bar{P}^k) = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^n |p_{ij}^k - \bar{p}_{ij}^k|, k = 1, \dots, m. \quad (8)$$

Based on such distances, the distance factor which measures the relative distance between P and $\bar{P}$, is defined as follows:

$$D(P, \bar{P}) = \sum_{k=1}^m d(P^k, \bar{P}^k). \quad (9)$$

Similarly, suppose that a CRP model produces the agreed solution as $\hat{P} = (\hat{P}^1, \hat{P}^2, \dots, \hat{P}^m)$. The distance between P^k and $\hat{P}^k$ can be computed as

$$d(P^k, \hat{P}^k) = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^n |p_{ij}^k - \hat{p}_{ij}^k|, k = 1, \dots, m. \quad (10)$$

and

$$D(P, \hat{P}) = \sum_{k=1}^m d(P^k, \hat{P}^k). \quad (11)$$

To evaluate the good performance of a CRP, $\hat{P}$, its solution is compared with the solution provided by $\bar{P}$ by means of the cost metric, $\phi(\hat{P}, \bar{P})$, which is defined as follows:

$$\phi(\hat{P}, \bar{P}) = \begin{cases} 1 - \frac{D(P, \hat{P})}{D(P, \bar{P})}, & \text{if } D(P, \hat{P}) \leq D(P, \bar{P}) \\ \frac{D(P, \bar{P})}{D(P, \hat{P})} - 1, & \text{if } D(P, \hat{P}) > D(P, \bar{P}). \end{cases} \quad (12)$$

We investigate properties of the previous metric.

- 1) If $D(P, \hat{P}) \leq D(P, \bar{P})$, then $0 \leq \phi(\hat{P}, \bar{P}) \leq 1$.
 - If $\hat{P} = \bar{P}$, then $\phi(\hat{P}, \bar{P}) = 0$, which means that $\hat{P}$ provides the minimum cost solution to reach consensus.
 - If $\hat{P} = P$, then $\phi(\hat{P}, \bar{P}) = 1$, which means that $\hat{P}$ provides the worst solution since we assume initial opinions are not under consensus, otherwise, make no sense to apply CRP.
 - If $\phi(\hat{P}, \bar{P})$ increases from $D(P, P)$ to $D(P, \bar{P})$, then $\phi(\hat{P}, \bar{P})$ decreases from 1 to 0.
- 2) If $D(P, \hat{P}) > D(P, \bar{P})$, then $-1 \leq \phi(\hat{P}, \bar{P}) \leq 0$. This case means that, costly changes have been made in the experts' preferences and thus, there is an excessive cost to achieve the consensus regarding the MCC model.
 - If $\hat{P} \rightarrow \bar{P}$, then $\phi(\hat{P}, \bar{P}) \rightarrow 0$, which also means that $\hat{P}$ provides the minimum cost solution to reach consensus. From this result, we can see that the metric $\phi(\hat{P}, \bar{P})$ is continuous at the point $\hat{P} = \bar{P}$.
 - If $D(P, \hat{P}) \rightarrow +\infty$, then $\phi(\hat{P}, \bar{P}) \rightarrow -1$, which means that $\hat{P}$ provides also the worst solution since there needs infinite cost.
 - If $D(P, \hat{P})$ increases from $D(P, \bar{P})$ to $+\infty$, then $\phi(\hat{P}, \bar{P})$ decreases from 0 to -1 .

From the previous analysis, we know that $\phi(\hat{P}, \bar{P}) \in [-1, 1]$, and when $\hat{P} = \bar{P}$, the CRP solution $\hat{P}$ is the best. The CRP solution $\hat{P}$ becomes worse when $\hat{P}$ goes far away from both sides of $\bar{P}$. Hence, the metric allows to compare the relative closeness of a CRP model to a MCC model.

We provide an example to show the method to evaluate the performance of a CRP model.

Example 3. Suppose that three experts e_1, e_2, e_3 with weights $W = (0.3, 0.4, 0.3)$ and costs $(c_1, c_2, c_3) = (1, 1, 1)$, respectively, provide their assessments over three alternatives in form of reciprocal FPRs $P = (P^k)_{3 \times 3}, k = 1, 2, 3$:

$$P^1 = \begin{pmatrix} 0.5 & 0.14 & 0.06 \\ & 0.5 & 0.27 \\ & & 0.5 \end{pmatrix}, P^2 = \begin{pmatrix} 0.5 & 0.55 & 0.13 \\ & 0.5 & 0.11 \\ & & 0.5 \end{pmatrix},$$

$$P^3 = \begin{pmatrix} 0.5 & 0.8 & 0.6 \\ & 0.5 & 0.27 \\ & & 0.5 \end{pmatrix}.$$

We consider the CRP model that is going to be evaluated $\hat{P}$ and the MCC model $\bar{P}$.

The first CRP model produces the following adjusted FPRs $\hat{P} = (\hat{P}^k)_{3 \times 3}, k = 1, 2, 3$:

$$\hat{P}^1 = \begin{pmatrix} 0.5 & 0.26 & 0.13 \\ & 0.5 & 0.22 \\ & & 0.5 \end{pmatrix}, \hat{P}^2 = \begin{pmatrix} 0.5 & 0.55 & 0.13 \\ & 0.5 & 0.11 \\ & & 0.5 \end{pmatrix},$$

$$\hat{P}^3 = \begin{pmatrix} 0.5 & 0.68 & 0.48 \\ & 0.5 & 0.27 \\ & & 0.5 \end{pmatrix}.$$

The MCC model produces the following adjusted FPRs $\bar{P} = (\bar{P}^k)_{3 \times 3}, k = 1, 2, 3$:

$$\bar{P}^1 = \begin{pmatrix} 0.5 & 0.5 & 0.11 \\ & 0.5 & 0.26 \\ & & 0.5 \end{pmatrix}, \bar{P}^2 = \begin{pmatrix} 0.5 & 0.56 & 0.13 \\ & 0.5 & 0.2 \\ & & 0.5 \end{pmatrix},$$

$$\bar{P}^3 = \begin{pmatrix} 0.5 & 0.57 & 0.18 \\ & 0.5 & 0.26 \\ & & 0.5 \end{pmatrix}.$$

We want to measure relative closeness of the first CRP model to the MCC model.

We first obtain that

$$(d(P^1, \hat{P}^1), d(P^2, \hat{P}^2), d(P^3, \hat{P}^3)) = (0.08, 0.0, 0.08).$$

and

$$(d(P^1, \bar{P}^1), d(P^2, \bar{P}^2), d(P^3, \bar{P}^3)) = (0.14, 0.033, 0.22).$$

We then obtain that

$$D(P, \hat{P}) = \sum_{k=1}^3 d(P^k, \hat{P}^k) = 0.16,$$

and

$$D(P, \bar{P}) = \sum_{k=1}^3 d(P^k, \bar{P}^k) = 0.393.$$

As a result the cost metric is computed as

$$\phi(\hat{P}, \bar{P}) = 1 - \frac{0.16}{0.393} = 0.5929. \quad (13)$$

Therefore, the CCM shows that the CRP provides a solution, $\hat{P}$, in which not all experts reach enough consensus. In contrast to the minimum cost solution, $\bar{P}$, in which all experts are close to the overall agreement, the CRP's solution $\hat{P}$ compensates experts with low degree of agreement with others with high agreement. From our view, such compensatory models should be further penalized because they are not obtaining genuine agreement, therefore, we further study this penalization in our CCM to carry out a proper analysis of the analysed CRP.

4.2. Amplified Cost Consensus Metric

Previous section introduces a novel CCM which allows to evaluate the performance of CRPs. However, Labella et al. shown in Labella et al. (2018) that many of such CRP models are compensatory and their solutions did not fulfil the constraint related to the parameter ε . In our opinion, the metric should apply a greater penalization to these models to reflect such a shortcoming from agreement point of view. Therefore, motivated by the method to amplify extreme values Yager and Petry (2014), we construct an ACCM (Amplified CCM) which can amplify extreme values of an expert. We adopt the amplification factor $f(x): [0, 1] \rightarrow [1, \infty)$, which satisfies $f(0) = 1$ and $f(a) \geq f(b)$ for $a > b$. In the following we present several forms of f :

$$f(x) = 1 + \tan\left(\frac{\pi}{2}x\right); \quad (14)$$

$$f(x) = \frac{\exp(x)}{1-x}; \quad (15)$$

$$f(x) = \frac{e-1}{e-\exp(x)}; \quad (16)$$

$$f(x) = \frac{1+kx}{1-x}, k \geq 0. \quad (17)$$

Table 5
Consensus models' parameters.

Wu and Xu (2012)	Zhang et al. (2012)	Herrera-Viedma et al. (2002)	Chiclana et al. (2008)	Kacprzyk and Zadrozny (2010)	Rodríguez et al. (2018)
$\alpha = 0.85$	$\alpha = 0.85$	$\alpha = 0.85$	$\alpha = 0.85$	$\alpha = 0.85$	$\alpha = 0.85$
$\beta = 0.8$	$cl = 0.95$	$\beta = 0.8$	$B = 0.8$	$\mu = 0.2$	$\omega = 1.5$
$\bar{CI} = 0.15$	$ccl = 0.85$	Aggregation quantifier = F_{most}	$\theta_1 = 0.7$	Aggregation quantifier = F_{most}	$\delta = 0.7$
$w_i = \frac{1}{m}, i \in \{1, \dots, m\}$		Exploitation quantifier = $F_{as \text{ many as possible}}$	$\theta_2 = 0.8$		$a = 1.0, b = 2$

The amplification factor of the expert e_k is then defined as

$$\bar{u}_k = f\left(d(P^k, \bar{P}^k)\right). \quad (18)$$

Based on amplification factor, the distance factor which measures the relative distance between P and $\bar{P}$, is defined as follows:

$$D_A(P, \bar{P}) = \sum_{k=1}^m \bar{u}_k \cdot d(P^k, \bar{P}^k). \quad (19)$$

Similarly to the previous section, suppose that a CRP model produces the agreed solution as $\hat{P} = (\hat{P}^1, \hat{P}^2, \dots, \hat{P}^m)$. The amplification factors are $\hat{u}_1, \hat{u}_2, \dots, \hat{u}_m$, and the distance factor is

$$D_A(P, \hat{P}) = \sum_{k=1}^m \hat{u}_k \cdot d(P^k, \hat{P}^k), \quad (20)$$

where $\hat{u}_k = f(d(P^k, \hat{P}^k))$, $k = 1, 2, \dots, m$.

To evaluate the good performance of a CRP, $\hat{P}$, its solution is compared with the solution provided by $\bar{P}$ by means of the cost metric, $\phi(\hat{P}, \bar{P})$, which is defined as follows:

$$\phi(\hat{P}, \bar{P}) = \begin{cases} 1 - \frac{D_A(P, \hat{P})}{D_A(P, \bar{P})}, & \text{if } D_A(P, \hat{P}) \leq D_A(P, \bar{P}) \\ \frac{D_A(P, \bar{P})}{D_A(P, \hat{P})} - 1, & \text{if } D_A(P, \hat{P}) > D_A(P, \bar{P}). \end{cases} \quad (21)$$

The properties of the ACCM can be inferred from the CCM definition in the previous section.

Example 4. We solve Example 3 here but applying amplification factors for the different experts:

In order to calculate the amplification factor of experts, we set

$$f(x) = \frac{1 + 1000x}{1 - x}.$$

Based on $d(P^k, \hat{P}^k)$, the amplification factors of experts e^k , $k = 1, 2, 3$ are obtained as

$$(\hat{u}_1, \hat{u}_2, \hat{u}_3) = (88.0434, 1.0, 88.0434).$$

Based on $d(P^k, \bar{P}^k)$, the amplification factors of experts e^k , $k = 1, 2, 3$ are obtained as

$$(\bar{u}_1, \bar{u}_2, \bar{u}_3) = (163.9535, 35.5172, 283.3333).$$

We then obtain that

$$D_A(P, \hat{P}) = \sum_{k=1}^3 \hat{u}_k \cdot d(P^k, \hat{P}^k) = 14.0870,$$

and

$$D_A(P, \bar{P}) = \sum_{k=1}^3 \bar{u}_k \cdot d(P^k, \bar{P}^k) = 86.4708.$$

As a result the cost metric is computed as

$$\phi(\hat{P}, \bar{P}) = 1 - \frac{14.0870}{86.4708} = 0.8371. \quad (22)$$

From this example and comparing with the results obtained in Example 3, we can see that the ACCM evaluates a CRP model by using the MCC model that considers the distances between FPRs,

Table 6
Behaviour default values.

Parameter	Value
p	1.0
Change	0.05

Table 7
Minimum distance cost metric parameters.

Parameter	Value
ε	$\varepsilon \in \{0.05, 0.06, 0.1, 0.15, 0.2, 0.3\}$
α	0.85
Cost	$c_i = 1, i \in \{1, \dots, m\}$
Weights	$w_i = \frac{1}{m}, i \in \{1, \dots, m\}$

and the importance of each distance, by multiplying each distance with its amplification effect. In such a way those models that compensate experts' opinions to achieve a 'not genuine' agreement are more penalized.

Remark 4. It is noted that any of the functions Eqs. (14)–(17) can be selected to define amplification factors. Here we have selected Eq. (17) with $k = 1000$ since it has a clear amplification effect for small variance of x .

5. Comparative study on the performances of consensus models

This section presents a fair comparative study on the performances of classical consensus models based on the metric proposed in previous section (see Eq. (21)). First, several representative consensus models are selected. Second, the comparative scenarios are described. Afterwards, a simulation process based on AFRYCA 3.0 (Labella et al., 2018; Palomares et al., 2014a) is carried out together with an analysis of the results obtained for each consensus model. Finally, a comparative analysis among all models is also performed.

5.1. Choosing consensus models

Several proposals of CRPs in GDM have been introduced in the specialized literature. For this reason, to carry out our study, a selection of several consensus models is done. Such a selection is composed to a greater extent of classical consensus models and, a more recent consensus model with the aim of carrying out a comparative analysis as complete and diverse as possible. This selection is based on the taxonomy reviewed in Section 1 and graphically represented in Fig. 1. Taking into account the taxonomy, the consensus models selection is divided into 2 groups: consensus models with and without feedback process, that work with the same type of preference relation, FPR. The consensus models selected based on both groups are:

- *Representative models with feedback process:* the selected models are that proposed by Herrera-Viedma et al. (2002), Chiclana et al. (2008) and Kacprzyk and Zadrozny (2010) and Rodríguez et al (Rodríguez et al. (2018)).

Table 8

Simulation results of CRPs with feedback process.

	Initial consensus degree	Final consensus degree	Number of rounds	Ranking	Solution
Herrera-Viedma et al. (2002)	0.75	0.88	3	$x_2 > x_4 > x_3 > x_1$	x_2
Chiclana et al. (2008)	0.75	0.85	12	$x_2 > x_4 > x_3 > x_1$	x_2
Kacprzyk and Zadrozny (2010)	0.75	0.88	9	$x_4 > x_2 > x_3 > x_1$	x_4
Rodríguez et al. (2018)	0.59	0.85	11	$x_4 > x_2 > x_1 > x_3$	x_4

Table 9

Simulation results of CRPs without feedback process.

	Initial consensus degree	Final consensus degree	Number of rounds	Ranking	Solution
Wu and Xu (2012)	0.75	0.86	16	$x_2 > x_4 > x_3 > x_1$	x_2
Zhang et al. (2012)	0.75	0.85	1	$x_2 > x_4 > x_3 > x_1$	x_2

- *Representative models without feedback process*: the selected models are that proposed by Wu and Xu (2012) and Zhang et al. (2012).

Remark 5. A brief description of the representative consensus models is provided in Appendix B.

5.2. Simulation scenarios

To evaluate the performance of the distinct consensus models selected in Section 5.1 by means of the proposed metric, it is necessary to define the conditions in which the simulations will be carried out. Such conditions will be determined by: (i) maximum number of rounds and consensus threshold in the CRP, (ii) the consensus models' parameters configuration (see Table 5), (iii) the experts' behaviour configuration (see Table 6), and (iv) the metric's parameters configuration (see Table 7).

As it was aforementioned, a CRP finishes when the minimum acceptable agreement, $\alpha \in [0, 1]$, or the maximum numbers of rounds allowed, *Maxround*, are reached, to avoid a never ending process. For the simulations, the predefined values assigned to α and *Maxround* are 0.85 and 30, respectively.

The selected consensus models use different parameters whose predefined values are represented in Table 5.

Thanks to AFRYCA 3.0 (Labella et al., 2018; Palomares et al., 2014a), it is possible to simulate the experts' behaviour. AFRYCA 3.0 includes components which simulates two different experts' behaviours: *standard behaviour* and *adverse behaviour*. The standard behaviour pattern simulates experts who can accept/refuse the suggestion received, otherwise the adverse behaviour simulates experts who can accept, refuse or defend suggestions. The selected behaviour to carry out the simulations is the standard behaviour which is simulated by a binomial probability distribution, configured by a parameter p that defines the probability for experts to accept suggestions. In order to carry out a proper comparison among the selected CRP models, the simulations in this contribution present an ideal scenario in which the experts always accept the suggestion thus, $p = 1$. When an expert accepts a suggestion, the change degree applied to the expert's preference is defined by another parameter noted as *change*. Both parameters are represented in Table 6.

To evaluate the CRP performance by means of the metric, it is necessary to configure its parameters: ε , α , *cost* and experts' weights. Such parameters are shown in Table 7.

5.3. Results and analysis

This section introduces an experimental study to evaluate and compare the performance of the selected classical consensus models.

Let us suppose a group of 8 students, $E = \{e_1, \dots, e_8\}$. The students plan to organize an end-of-year trip, thus they must

Table 10ACCM results ($\alpha = 0.85$).

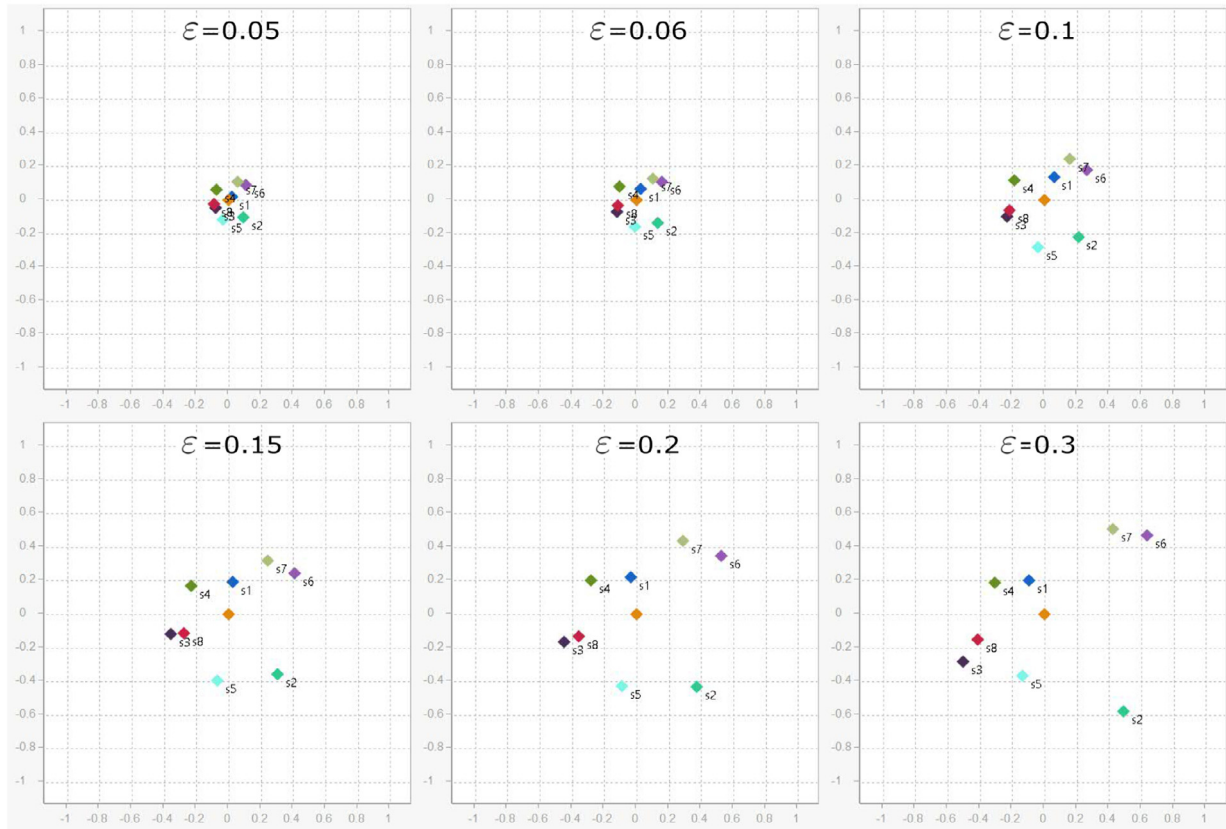
Consensus model	ε	ACCM
Herrera-Viedma et al. (2002)	0.05	0.73
	0.06	0.63
	0.1	0.45
	0.15	0.15
	0.2	-0.29
Chiclana et al. (2008)	0.3	-0.6
	0.05	0.67
	0.06	0.61
	0.1	0.64
	0.15	0.63
Kacprzyk and Zadrozny (2010)	0.2	0.62
	0.3	0.64
	0.05	0.68
	0.06	0.62
	0.1	0.65
Rodríguez et al. (2018)	0.15	0.64
	0.2	0.64
	0.3	0.66
	0.05	0.53
	0.06	0.35
Wu and Xu (2012)	0.1	0.04
	0.15	-0.32
	0.2	-0.59
	0.3	-0.77
	0.05	0.7
Zhang et al. Zhang et al. (2012)	0.06	0.59
	0.1	0.4
	0.15	0.07
	0.2	-0.35
	0.3	-0.63
	0.05	0.77
	0.06	0.72
	0.1	0.74
	0.15	0.74
	0.2	0.73
	0.3	0.75

make a common decision about choosing the city to travel among 4 possible alternatives, $X = \{x_1, \dots, x_4\}$, which include: Athens, Portsmouth, Belfast and Chengdu. All students express their preferences as FPRs which are generated by AFRYCA. The corresponding data set is available in the AFRYCA website¹. The simulations are carried out under the conditions predefined in Section 5.2 and the results are shown below in Tables 8 and 9.

Once the results of each consensus model have been obtained, the evaluation of their performances is carried out by the ACCM (see Table 10). Note that, in order to evaluate the solutions of the representative consensus models with the solutions of the MCC models, those which use a consensus measure based on the

¹ <http://sinbad2.ujaen.es/afryca/>.

(a) MCC Model (M-6)



(b) MCC Model (M-7)

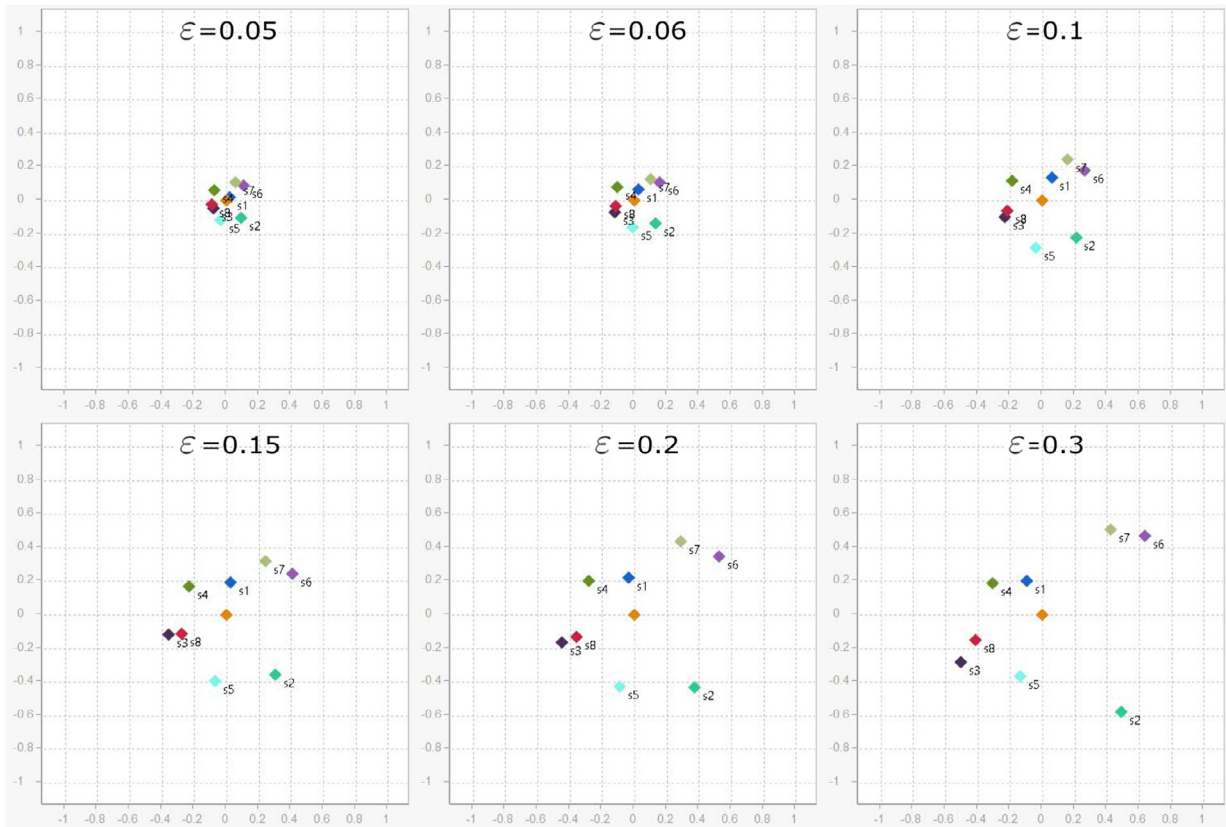


Fig. 3. Visualization of the models (M-6) and (M-7) with different ε .

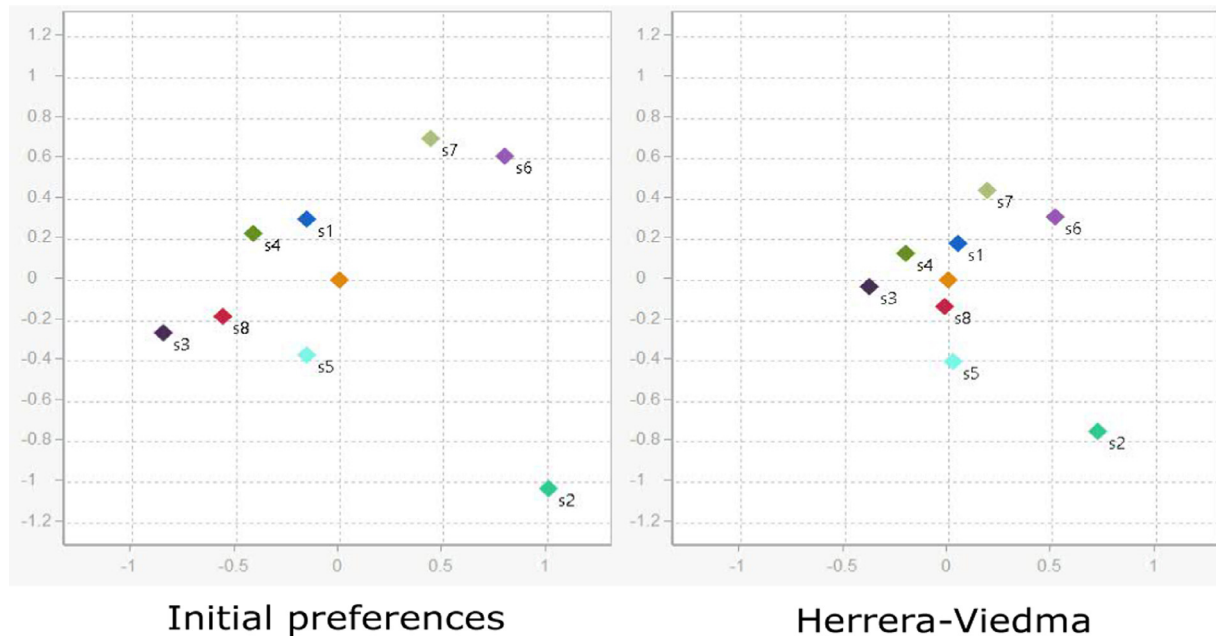


Fig. 4. The final round of Herrera-Viedma et al.'s model.

distance between experts and the collective opinion have been compared with the MCC model (M-6) and those whose consensus measure is based on the distance among experts have been compared with the MCC model (M-7).

5.3.1. Analysis for each representative model

Here a separate analysis for each consensus model according to the ACCM is given. Such an analysis consists of a brief explanation of the results obtained with their graphical visualization together with an analysis of its performance. Furthermore, the graphical visualization of the solutions provided by the MCC models with different values of ε and $\alpha = 0.85$ are also presented in Fig. 3.

- Herrera-Viedma et al.'s model (Herrera-Viedma et al., 2002)
Fig. 4 shows that several experts, for instance, s_2 , are far from the rest of the experts, by obtaining a solution in which experts are slightly dispersed. This can also be seen from Table 10. The ACCM shows that the consensus model solution is far from the minimum cost solution with the more restrictive values of ε , such as 0.05, 0.06 or 0.1, as can be appreciated in Figs. 3(a) and 4. The closest solution respect to the minimum cost is provided when $\varepsilon = 0.15$. From $\varepsilon = 0.2$, the bigger ε , the higher the excessive cost. To conclude the analysis, note that it is evident that the consensus model reaches the agreement among experts very fast, but, at the same time, the ACCM shows that the experts do not reach enough agreement degree among them. This situation happens due to experts' consensus degree on each alternative is based on an average operator. Thus, those experts with a high level of agreement compensate those who are further from others.
- Chiclana et al.'s model (Chiclana et al., 2008)
As in the previous model, Fig. 5 shows that the opinions of some experts, for instance, s_2 , are far from the other experts. Once again, the ACCM corroborates this situation (Table 10). The experts do not reach enough agreement between them for any value of ε and the opinions of some experts are relatively far from each other, being this situation reflected in Fig. 3(b). Again, those experts with a high level of agreement compensate those who are further from others.
- Kacprzyk et al.'s model (Kacprzyk & Zadrożny, 2010)

Table 10 shows similar results to Chiclana et al.'s consensus model. Thus, the analysis of this consensus model can be inferred from Chiclana et al.'s model in a similar way.

- Rodríguez et al.'s model (Rodríguez et al., 2018)
Rodríguez et al.'s model presents a solution similar to the one provided by the minimum cost model in several values of ε according to the ACCM (see Table 10). Specifically, the model obtains a solution relatively nearby to the minimum cost model for the lowest values of ε , 0.05 and 0.06. In the case of, $\varepsilon = 0.1$, the solution is practically identical to the one provided by the minimum cost, being the closest one to the minimum cost solution among all the models for any value of ε . The similarity among both solutions can be easily appreciated by comparing Figs. 3 and 7. For the rest of ε values the solution present an excessive cost, whereas for $\varepsilon = 0.15$ the excessive cost is not too high, for $\varepsilon = 0.2$ and 0.3 the excessive cost is more accentuated and the solution is far from the minimum cost solution. Therefore, Rodríguez et al.'s model provides a solution in which experts' opinions are closer to each other and there is no any expert far from the opinions of the rest of the group. Furthermore, the experts' opinions are not drastically modified, particularly in certain ε values. However, in spite of obtaining a good solution from the minimum cost's point of view, the model needs 11 rounds to reach the consensus. This shows that the number of rounds is not a good criterion to evaluate the CRPs' performance.
- Wu et al.'s model (Wu & Xu, 2012)
The solution provided by Wu et al.'s model is closer to the minimum cost solutions according to Figs. 3(a) and 8, with several values of ε and it is also certified by the ACCM. By analysing the results with the ACCM (Table 10), we can appreciate that the solution provided by Wu et al.'s model is relatively close to the minimum cost solution when ε is equal to 0.1 and 0.2, the latter with an excessive cost. When $\varepsilon = 0.15$, the solution obtained is quite similar to the one provided by the MCC model. For the rest of the values of ε the solution is far from the minimum cost solution by presenting a high cost. To conclude the analysis, we can ensure that Wu et al.'s model provides a solution for specific values of ε in which the opinions of the experts are close to each other and their preferences have not

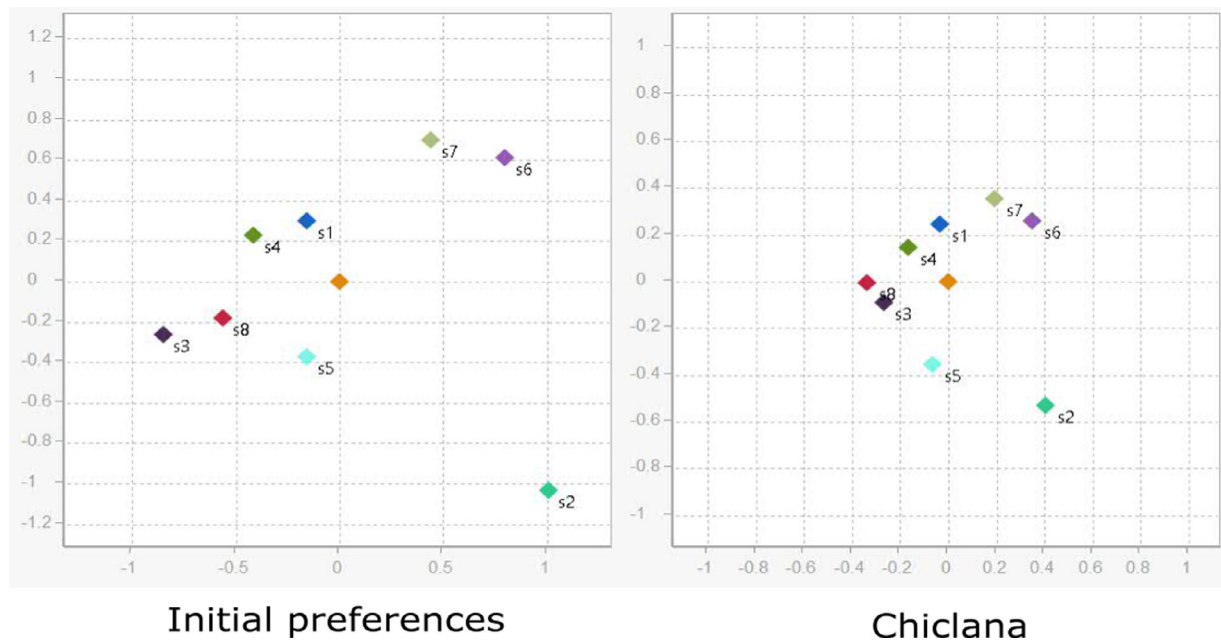


Fig. 5. The final round of Chiclana et al.'s model.

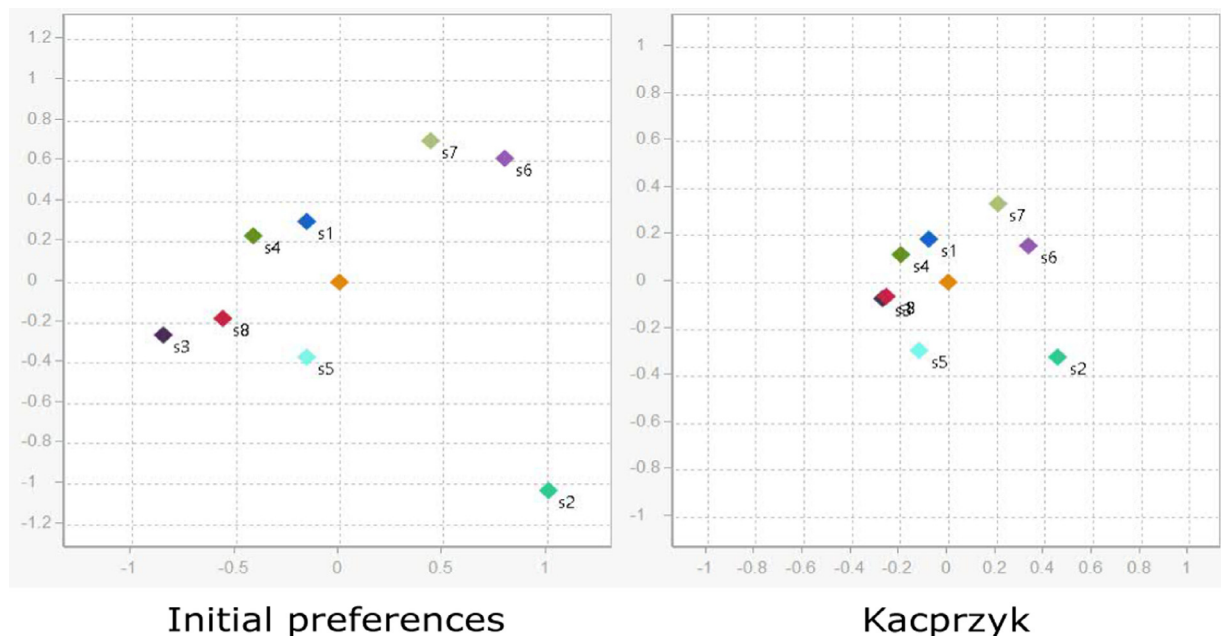


Fig. 6. The final round of Kacprzyk et al.'s model.

been changed excessively to reach consensus. Nevertheless, it should be noted that the consensus model needs 16 rounds to reach the consensus threshold level, due to the fact that just one expert's preference is changed in each round.

- Zhang et al.'s model (Zhang et al., 2012)

The results of the ACCM show that Zhang et al.'s model provides a solution far from the minimum cost solution with any value of ε . The ACCM shows a considerable distance between the experts' preferences provided by the representative consensus model and the minimum cost model, as it is shown in Figs. 3(b) and 9.

5.3.2. Global analysis

This section introduces a global analysis of the different representative consensus models according to the metric results.

A) Representative consensus models with feedback mechanism: Firstly, we will only take into consideration the consensus models with a feedback process, Herrera-Viedma et al.'s model, Chiclana et al.'s model, Kacprzyk et al.'s model and Rodríguez et al.'s model to do a global analysis. A classical analysis would consider the number of rounds necessary to reach consensus. Regarding this issue, it is evident that Herrera-Viedma's model provides the best performance. However, by analysing the results of the metric for the metric's model (M-6), we can see that Herrera-Viedma's model provides a solution in which experts are far from each other if we consider restrictive values of ε , which means that experts do not reach enough consensus between them. To compensating this situation, it changes some experts' preferences significantly. On the other hand, Herrera-Viedma

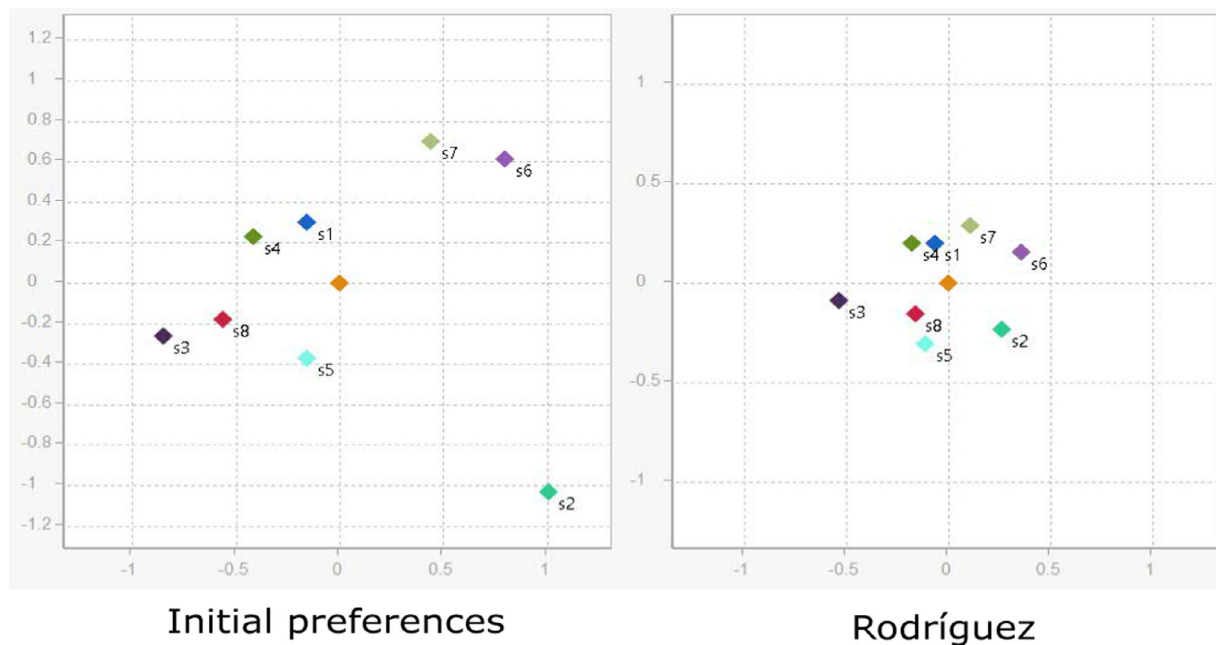


Fig. 7. The final round of Rodríguez et al.'s model.

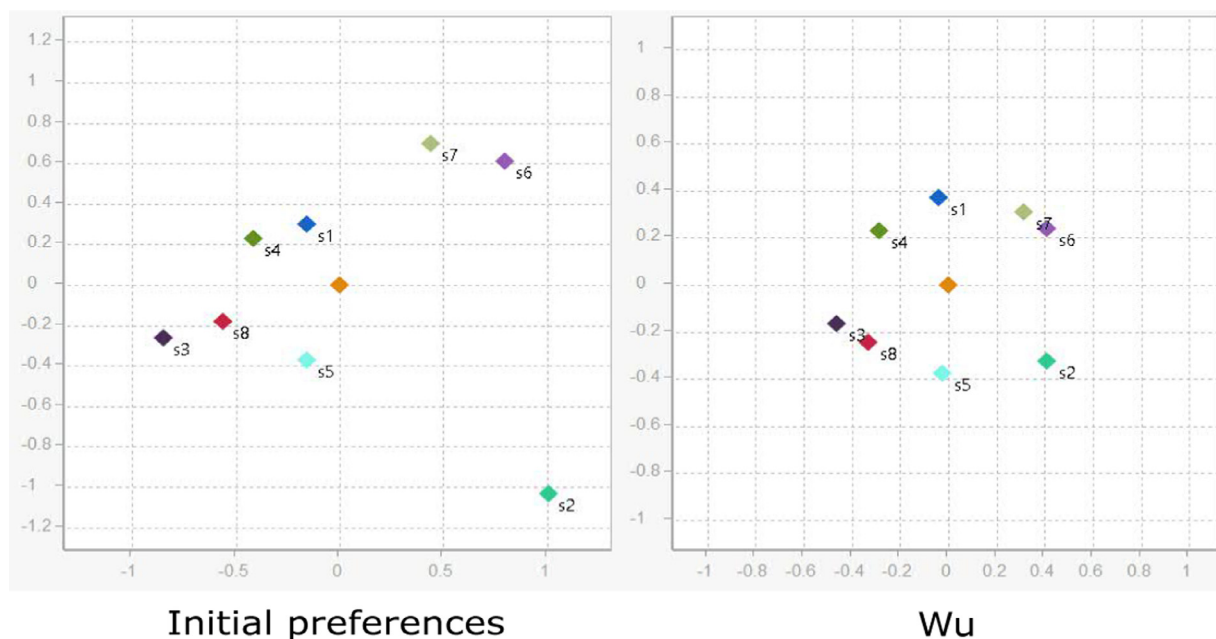


Fig. 8. The final round of Wu et al.'s model.

et al.'s model solution is close to the minimum cost solution for some values of ε , which means that the experts, in this case, are not so far from each other and their preferences have not been modified so much compared with the previous consensus models. Chiclana et al.'s model provides a worse solution than Herrera-Viedma's model for almost any value of ε , thus, once again the experts do not achieve enough consensus among them. Kacprzyk et al.'s model provides similar results to Chiclana's model but it needs less rounds to reach the consensus threshold. However, unlike the rest of the models, Rodríguez et al.'s model does provide a solution in which experts' preferences are closer to each other, even for restrictive values of ε , which means that the solution obtained is the closest to the genuine agreement. To conclude, all the analyzed consensus models except Ro-

dríguez et al.'s model present the same drawback, they provide a compensated solution in which experts with a high level of agreement 'hide' those experts who do not reach enough consensus and thus, it is not obtained a genuine agreement. The facts presented in this analysis prove that evaluating the number of rounds necessary to reach consensus is not enough to carry out a proper analysis of the performance of a consensus model.

B) Representative consensus models without feedback mechanism:

Afterwards, we consider the consensus models without a feedback process, Wu et al.'s model and Zhang et al.'s model. There is no doubt that Wu et al.'s model provides a much better solution than Zhang's model since the former presents a solution closer to the minimum cost solution for

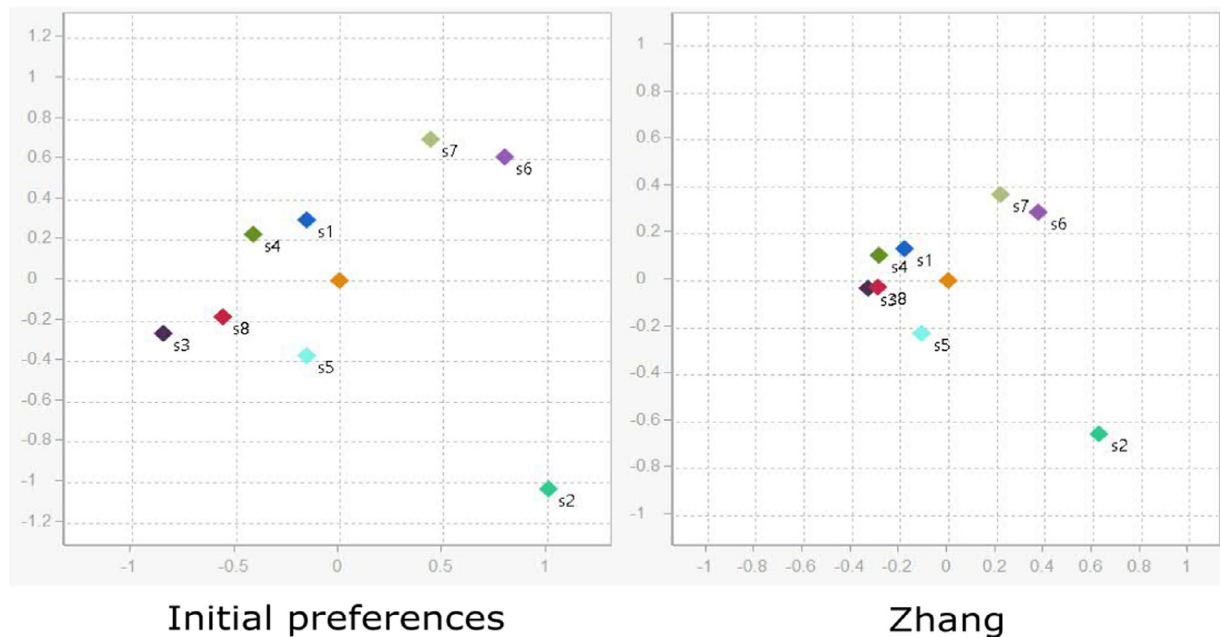


Fig. 9. The final round of Zhang et al.'s model.

several values of ε , which means that experts are close to each other and the modifications of their preferences are not excessive. Note that the number of rounds necessary to reach consensus is greater in Wu et al.'s model than Zhang's model since Zhang's model uses a linear programming model.

C) Both:

Finally, by comparing globally the models regarding the new metric presented, in spite of Zhang et al.'s consensus model is the fastest to reach the consensus, it provides the worst solution for any value of ε regarding the other consensus models. The reason for its less number of rounds is that this model does not use a feedback process and thus, does not consider the participation of the experts to change their opinions but they are modified directly by using a linear programming model. The linear programming model is executed only once, by obtaining the modified experts' preferences in just one round, which does not imply that better solutions are reached, as can be seen in our analysis. Herrera-Viedma's model also reaches consensus in a few number of rounds and, in addition, it presents a feedback mechanism in which experts' opinions are taken into account for changing their preferences. Nevertheless, it has been demonstrated that such a model presents a solution in which experts with a high level of agreement compensate those who are further from others. Furthermore, such a situation is not exclusive of this model, since each analysed consensus model presents the same situation. The model that provides a more homogeneous solution for several values of ε , in which the experts are close to each other and the compensation is not so evident, is Rodríguez et al.'s model, despite being one of the consensus model that needs more rounds to reach consensus.

6. Conclusions

Nowadays, consensual decisions are increasingly important in decision making problems in which it is important to remove the disagreement among experts to obtain a better solution that is more appreciated by the group, giving rise to the CRPs. Due to this

fact, there are many proposals on CRPs, but there is no any suitable criterion to evaluate and compare the performance of CRPs.

A novel cost metric to compare the performance of CRPs has been introduced. The metric is based on two novel MCC models that consider the distance of the experts to the collective opinion and also guarantee a minimum agreement among experts and thus, an acceptable and better level of consensus is obtained. The obtained results show that the new metric can be effectively used to make comparison between CRPs, since it allows to reveal anomalous situations in their performance, such as the compensation situations, that cannot be detected by using other criteria. This metric has been implemented and integrated in a decision support system.

In the future research, we will study how to design more efficient MCC models with large-scale GDM problems, and their application in real world problems such as, business management, political negotiation, etc. With the boom of research on CRP in social network, the comparison analysis of these CRP models is an emerging field which deserves to investigate. The proposed cost metric might be one useful criterion for such comparison.

Declarations of Competing interest

None.

Acknowledgments

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Appendix A. MCC models considering non-normalized values

[Section 3.1](#) presents MCC models that take into account level of agreement and distance to collective opinion in which the experts' preferences are provided by numerical values. Nevertheless, these models consider exclusively values valued in $[0, 1]$. To apply these

MCC models to problems in which the original values are not valued in $[0, 1]$, a normalization process is defined as follows:

$$o'_i = \frac{o_i - \min_{1 \leq i \leq m} \{o_i\}}{\max_{1 \leq i \leq m} \{o_i\} - \min_{1 \leq i \leq m} \{o_i\}}, i = 1, 2, \dots, m \quad (\text{A.1})$$

Thus, the model (M-4) can be transformed into the following one:

(M-4)(b)

$$\begin{aligned} \min & \sum_{i=1}^m c_i |\bar{o}'_i - o'_i| \\ \text{s.t.} & \begin{cases} \bar{o} = \sum_{i=1}^m w_i \bar{o}'_i \\ |\bar{o}'_i - \bar{o}| \leq \varepsilon, i = 1, 2, \dots, m \\ \sum_{i=1}^m w_i |\bar{o}'_i - \bar{o}| \leq \gamma. \end{cases} \end{aligned}$$

In the same way, the model (M-5) can be transformed into the following one:

(M-5)(b)

$$\begin{aligned} \min & \sum_{i=1}^m c_i |\bar{o}'_i - o'_i| \\ \text{s.t.} & \begin{cases} \bar{o} = \sum_{i=1}^m w_i \bar{o}'_i \\ |\bar{o}'_i - \bar{o}| \leq \varepsilon, i = 1, 2, \dots, m \\ \sum_{i=1}^{m-1} \sum_{j=i+1}^m \frac{w_i + w_j}{m-1} |\bar{o}'_i - \bar{o}'_j| \leq \gamma. \end{cases} \end{aligned}$$

Although the original values have been normalized into $[0, 1]$ and consequently the adjusted values $\bar{o}'_i \in [0, 1]$, the latter can be transformed into the range of the original values by using

$$\bar{o}_i = \left(\max_{1 \leq i \leq m} \{o_i\} - \min_{1 \leq i \leq m} \{o_i\} \right) \bar{o}'_i + \min_{1 \leq i \leq m} \{o_i\}, i = 1, 2, \dots, m. \quad (\text{A.2})$$

Finally, the resulting cost obtained from the normalized values can also be transformed according to the values in the original range

$$\text{Cost} = \sum_{i=1}^m c_i |\bar{o}_i - o_i| = \left(\max_{1 \leq i \leq m} \{o_i\} - \min_{1 \leq i \leq m} \{o_i\} \right) \text{Cost}'. \quad (\text{A.3})$$

where Cost' is the cost obtained from the normalized values.

Appendix B. Description of the representative consensus models

A brief description of the selected representative consensus models is introduced below.

- *Herrera-Viedma et al.'s model (Herrera-Viedma et al., 2002)*: this model follows the *soft consensus* view [Herrera-Viedma et al. \(2014\)](#) and uses proximity measures provided by a moderator. The measures of consensus and proximity are computed through the comparison between the individual experts' preferences and the collective solution. Furthermore, a comparison for alternatives in each consensus round is carried out, computing the current consensus in each moment during the CRP. Another relevant aspect in this model is, which is able to manage distinct preferences relations, unifying all of them into FPR. The parameters of Herrera-Viedma et al.'s consensus model are briefly introduced here (see [Herrera-Viedma et al. \(2014\)](#) for further detailed descriptions):

- β : parameter related to the control of the OR-LIKE aggregation operator that computes the global consensus degree.
- Aggregation quantifiers: parameters related to the linguistic quantifier used to compute the collective preference by means of the OWA operator.
- Exploitation quantifiers: parameters related to the linguistic quantifier used to compute dominance and non-dominance degrees and conduct preferences of experts into preference orderings.
- *Chiclana et al.'s model (Chiclana et al., 2008)*: this model integrates individual consistency for the experts' preferences and the consensus measure is based on the computation between pairwise similarities. Several relevant models such as [Dong, Zhang, Hong, and Xu \(2010\)](#) and [Zhang et al. \(2012\)](#) are based on this model. The parameters of Chiclana et al.'s consensus model are briefly introduced here (see [Chiclana et al. \(2008\)](#) for further detailed descriptions):
 - B: parameter related to the consistency threshold for preferences.
 - θ_1 : parameter related to the low consensus threshold. If consensus degree is lower than this value, a low consensus preference search is applied.
 - θ_2 : parameter related to the medium consensus threshold. If consensus degree is lower or higher than this value, a medium or high consensus level is applied, respectively.
- *Kacprzyk et al.'s model (Kacprzyk & Zadrozny, 2010)*: this model is based on the notion of *soft consensus* under fuzzy preference relations. Similarities between pair of experts are computed at level of assessments, as alpha-degrees of sufficient agreement. Distinct consensus degrees are obtained at different levels from such similarities, based on quantifier-guided OWA aggregation. The parameters of Kacprzyk et al.'s consensus model are briefly introduced here (see [Kacprzyk and Zadrozny \(2010\)](#) for further detailed descriptions):
 - μ : parameter related to the non-strict similarity between experts' preferences.
 - Aggregation quantifiers: parameters related to the linguistic quantifier used to compute the collective preference by means of the OWA operator.

- *Wu et al.'s model (Wu & Xu, 2012)*: this model deals with individual consistency and computes the consensus measures based on the distance between the individual experts' preferences and the collective opinions. This model is relatively easy and straightforward and guarantees the acceptable consistency ([Wu & Xu, 2012](#)) for each individual preference and the whole group in the CRP. The parameters of Wu et al.'s consensus model are briefly introduced here (see [Wu and Xu \(2012\)](#) for further detailed descriptions):
 - CI: parameter related to the individual consensus threshold.
 - β : parameter related to the update coefficients for the preferences.
 - W_i : parameter related to the experts' weights.
- *Zhang et al.'s model (Zhang et al., 2012)*: this model preserves the initial preference information and extends the consistency-driven model proposed by Chiclana et al. to guarantee the minimum cost of modifying preferences. Aside from guiding the CRP, this model allows to achieve a high level of consistency for each individual preference relation. The parameters of Zhang et al.'s consensus model are briefly introduced here (see [Zhang et al. \(2012\)](#) for further detailed descriptions):
 - cl: parameter related to the minimal consistency level that each individual preferences have to reach.
 - ccl: parameter related to the minimal consistency level that the different preferences have to reach.

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