A robust Power Flow Algorithm Based on Bulirsch-Stoer Method

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***Abstract*— In this paper, we address the Load-Flow (LF) problem of very large-scale systems. This type of systems shows a very narrow Region of Attraction and most of LF solvers tend to fail when a flat initial guess point is used. On the other hand, the solution of these systems frequently involves very large matrices and vectors. Consequently, a robust LF method must be used to find the correct solution of these systems. This paper proposes a robust and efficient LF solver based on the Bulirsch-Stoer algorithm. Moreover, a simple modification is proposed in order to improve its computational performance. The proposed methods are tested using various very large-scale systems (i.e. more than 3000 buses) and compared with several standard and robust LF techniques. The obtained results show that the proposed methods are more suitable for solving the LF problem of very large-scale systems.**

Index Terms— **Robust power flow, initial guess point, Bulirsch-Stoer, large-scale power systems.**

I. Introduction

Load-Flow (LF) is likely the most important tool in planning and operation of power systems [1, 2]. Owing to its importance, LF techniques must be constantly updated with the aim to tackle the challenges of the today's power system operation. In this sense, developing novel methodologies suitable for solving the LF in the supergrid paradigm is always justified and well-received.

From a mathematical point of view, a supergrid should be considered as a very large-scale system [3, 4]. This type of systems presents several challenges for most of state-of-art LF techniques. They have a huge number of variables are often badly-initialized, where these systems show a narrow Region of Attraction [5, 6].

Newton-Raphson method (NR) is the most standard technique for solving the LF. Nevertheless, its performance becomes poor when the initial guess is not good enough.

The problem of badly-initialized LF cases has not been much investigated explicitly. An exception is the reference [7]. This approach consists in computing the initial voltage angles as a result of the well-known DC-Load-Flow [8]. Then, this information is used to compute the initial voltage magnitudes.

On the other hand, several robust LF techniques have been proposed in the literature over decades. Despite that these approaches are originally conceived for ill-conditioned systems (i.e. heavily loaded values and large R/X ratios), they occasionally show good performance in badly-initialized cases.

In [9], a robust LF solver based on the second-order formulation has been proposed. In this approach, a multiplier is computed during the iterative process as result of an optimization problem, which modifies the correction vector in order to avoid the divergence. This method rarely diverges. However, this method often employs many iterations to converge. Other robust LF methods have been developed in the literature [10-14].

The Continuous Newton's method (CN) was presented in [15] and, later, it was adapted for solving the LF in [5]. CN establishes an analogy between the LF and a set of autonomous ordinary differential equations (ODE). Owing to this analogy, any well-assessed numerical method can be adapted for solving the LF problem. In [5], the Explicit Euler method and the 4th order Runge-Kutta have been employed. These methods have been tested in a badly-initialized large-scale system. In this situation, Continuous Newton's approaches showed good performance.

A robust LF solver based on the Levenberg-Marquardt's method has been proposed in [16] and several high-order schemes have been developed in [17, 18]. These methods are quite robust with respect to the condition of the system. However, their robustness with respect to the initial guess point has not been proven yet. Furthermore, the convergence properties of the methods [17, 18] depend strongly of a set of predetermined parameters, which must be chosen carefully. In [16], the authors proved that the standard Levenberg-Marquardt's method may show some convergence difficulties (i.e. too much iterations and the correct solution is not always ensured).

Several solvers based on the Lyapunov theory have been presented in the literature [16, 19, 20]. These methods are based on stating the LF as a minimization problem. Then, a Lyapunov function is defined and the LF formulation is established as an artificial dynamic system. Therefore, the LF is solved by means of a numerical arrangement. These methods do not need to invert any matrix, however, its computational performance in large, and more significantly in very large-scale systems, may be poor [19].

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In [21], numerical polynomial homotopy continuation method has been proposed to find all load flow solutions. However, this methodology is not suitable for large scale problems. In [22], the power flow problem has been solved based on mixed complementarity approach. This approach is tested in a 2975-bus system. The obtained results have showed that this methodology needs relatively high computation time to solve the test system. In [23], the load flow problem of distribution networks has been solved using Wirtinger Calculus. Power flow problem over arbitrary networks has been solved in [24]. The suitability of this approach for large-scale systems has not been covered in this reference.

In conclusion, solving the LF problem of very large-scale systems demands very efficient and robust solvers. This paper proposes a robust and efficient LF solver based on the Bulirsch-Stoer algorithm. Moreover, a simple modification is proposed in order to improve its computational performance.

The proposed methods are tested using different very large-scale systems (i.e. more than 3000 buses). Their effectiveness has been validated by comparing their performance with other standard and robust LF techniques.

Remainder of this paper is organized as follows. Section II, introduces the LF problem, its formulation and a brief description of Continuous Newton's method. The BS method is outlined and adapted for solving the LF in Section III. In Section IV, a modification of the standard BS method is proposed. Some practical recommendations are given in Section V. Simulations carried out and results obtained are described and reported in Section VI. Finally, the main conclusions are summarized in Section VII.

II. Outlines of the LF Problem

The LF problem models the nonlinear relationships among the injected power at system buses, power demands, bus voltages and the circuit parameters. Its solution provides the voltage magnitude and angle at each bus, power flows and MVA loadings at both ends of transmission lines and transformers, and the total system losses. Firstly, let us consider the active and reactive power mismatches for each bus, which can be given as [25]:

$$\Delta P_i = P_i^{sch} - \sum_{j=1}^{n} |V_i| \, |V_j| |Y_{ij}| \cos(\theta_{ij} - \delta_i + \delta_j) \tag{1}$$
$$\Delta Q_i = Q_i^{sch} - \sum_{j=1}^{n} |V_i| \, |V_j| |Y_{ij}| \sin(\theta_{ij} - \delta_i + \delta_j) \tag{2}$$

where P_i^{sch} and Q_i^{sch} denote the injected active and reactive powers at bus i respectively, $V_i \angle \delta_i$ denote the complex voltage at bus i, $Y_{ij} \angle \theta_{ij}$ denote the ij^{th} element of admittance matrix and n is the total number of buses.

Voltage angles at PV buses and voltage angles and magnitudes at PQ buses are the variables of LF problem. Therefore, the LF state vector can be defined as:

$$\boldsymbol{x} = \left[\boldsymbol{\delta_{PV}} \cup \boldsymbol{\delta_{PQ}} \cup \boldsymbol{V_{PQ}}\right] \tag{3}$$

where, $\boldsymbol{\delta_{PV}}$ and $\boldsymbol{\delta_{PQ}}$ are the vectors of PV and PQ buses voltage angles respectively and $\boldsymbol{V_{PQ}}$ is the vector of PQ buses voltage magnitudes. In order to simplify the notation, compact version of (1) and (2) will be used onwards:

$$\mathbf{g}(\boldsymbol{x}) = 0 \tag{4}$$

Since (4) are nonlinear, an iterative technique must be used for solving them. NR could be considered the standard method for LF and widely applied in industry applications. A generic k^{th} NR iteration for solving LF problem can be defined as:

$$\begin{aligned} \Delta\boldsymbol{x}^{(k)} &= -\left[\mathbf{g}_{\mathbf{x}}^{(\mathrm{k})}\right]^{-1}\mathbf{g}^{(\mathrm{k})} \\ \boldsymbol{x}^{(k+1)} &= \boldsymbol{x}^{(k)} + \Delta\boldsymbol{x}^{(k)} \end{aligned} \tag{5}$$

where, $\mathbf{g_x}$ is the Jacobi matrix of the system, which is formed by partial derivatives of (4) with respect to the state vector (3) and the superscript k enumerates the current iteration. LF algorithm stops if the following condition is satisfied:

$$\|\mathbf{g}\|_\infty \leq \epsilon \tag{6}$$

where, ϵ is a preset convergence parameter which is typically taken smaller than 10^{-3}. The algorithm also stops if the number of iterations is greater than a given threshold ($k > k_{max}$). In this case, the algorithm has likely failed to converge.

- Brief description of the Continuous Newton's Method

The CN was introduced in [15] and it was successfully applied for solving the LF in [5]. The CN establishes an analogy between the LF and a set of ODEs. On the basis of this analogy, any well-assessed numerical method can be used for solving the LF. In this subsection, we described briefly this methodology. Firstly, let us consider a set of ODEs, as follows:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \tag{7}$$

The simplest method to solve (7) is the Explicit Euler method. The generic k^{th} iteration is computed as follows:

$$\begin{aligned} \Delta\boldsymbol{x}^{(k)} &= \Delta t \boldsymbol{f}\left(\boldsymbol{x}^{(k)}\right) \\ \boldsymbol{x}^{(k+1)} &= \boldsymbol{x}^{(k)} + \Delta\boldsymbol{x}^{(k)} \end{aligned} \tag{8}$$

where, Δt is the time step. An analogy between (5) and (8) is easily established if one defines:

$$\boldsymbol{f}(\boldsymbol{x}) = -\left[\mathbf{g}_{\mathbf{x}}^{(\mathrm{k})}\right]^{-1}\mathbf{g}^{(\mathrm{k})} \tag{9}$$

Therefore, (8) can be viewed as a NR iteration with $\Delta t = 1$. In [5], the author already applied the Explicit Euler method and the 4^{th} order Runge-Kutta formula for solving the LF. In [5], it was demonstrated that CN is asymptotically stable if the solution of (4) exists, and it is reachable if the initial guess is inside of the Region of attraction.

III. LF solution based on BS method

The BS algorithm [26, 27] (also called the Gragg-Bulirsch-Stoer algorithm [28]) is a powerful method for solving set of ODEs. In fact, it is believed that the BS is the best-known way to obtain high accuracy solutions to ODEs [29]. The BS collects the advantages of various numerical arrangements like the Explicit Euler method or the modified midpoint rule. The BS starts taking an arbitrary "big" step size H, which is splitted into n equal smaller steps as follows:

$$h = {}^{H}/_{n} \tag{10}$$

where, n is the desired number of divisions.

The normal BS procedure starts with a small value of n and increases it if the calculated state value is not accuracy enough.

Theoretically, the zero error is reached when $n = \infty$, it is just an idealization. Practically, two approaches have been proposed in the literature [26, 29, 30]:

$$\boldsymbol{n} = 2, 4, 6, 8, 12, 16, 24, 32, 48, 64 \ldots \tag{11}$$
$$\boldsymbol{n} = 2, 4, 6, 8, 10, 12, 14 \ldots \tag{12}$$

Theses series are endless. Therefore, it is necessary truncate them at a specified point. It is believed that the highest n implies the highest robustness. Nevertheless, the highest n also induces the highest computational effort. Fig. 1 shows a sketch of the BS procedure. As can be seen, more accuracy results are achieved by increasing n, however, more intermediate points need to be calculated. It is worth notice that the approximation with $n = \infty$ perfectly matches with the correct point at $t + H$.

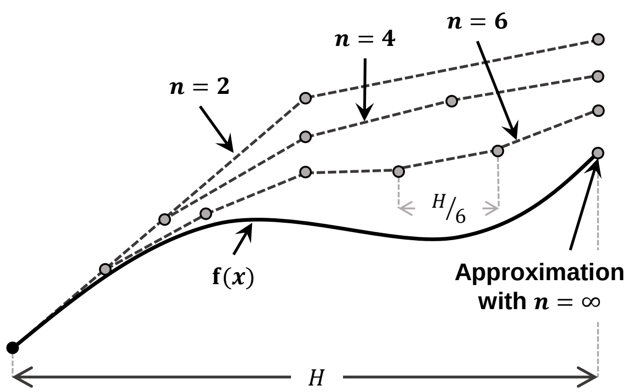

Fig 1 Sketch of the Bulirsch-Stoer method.

Following the CN's principle, the BS can be rightly adapted to solve the LF problem as follows:

- **Step 0**
- Input the system data and construct the Y_{BUS} matrix.
- Define ϵ and k_{MAX}
- Select an initial guess point $\boldsymbol{x}^{(0)} = \boldsymbol{z}_{\boldsymbol{0}}^{(0)}$
- Select the series which will be used (11) or (12).
- Define n_{MIN} and $n_{MAX} \geq n_{MIN}$ to truncate the series.
- Take $n = n_{MIN}$
- Define the "big" step size H
- Calculate the "small" step size using (10).
- Set $k = 0$
- **Step 1**
- Calculate the LF Jacobi matrix $\mathbf{g}_{\mathbf{x}}^{(\mathrm{k})}$ and the mismatches vector $\mathbf{g}^{(\mathrm{k})}$
- **Step 2**
- Calculate the NR correction vector using (13) and $\boldsymbol{z}_{\boldsymbol{1}}$ using the Explicit Euler method (14).

$$\Delta\boldsymbol{x}^{(k)} = -\left[\mathbf{g}_{\mathbf{x}}^{(\mathrm{k})}\right]^{-1}\mathbf{g}^{(\mathrm{k})} \tag{13}$$
$$\boldsymbol{z}_{\boldsymbol{1}}^{(k)} = \boldsymbol{x}^{(k)} + h\Delta\boldsymbol{x}^{(k)} \tag{14}$$

- Calculate the remainder intermediate points using the modified midpoint rule as:

$$\Delta\boldsymbol{z}_{\boldsymbol{m}}^{(k)} = -\left[\mathbf{g}_{z_m}^{(\mathrm{k})}\right]^{-1}\mathbf{g}(\boldsymbol{z}_{\boldsymbol{m}}^{(k)}) \tag{15}$$
$$\boldsymbol{z}_{\boldsymbol{m+1}}^{(k)} = \boldsymbol{z}_{\boldsymbol{m-1}}^{(k)} + 2h\Delta\boldsymbol{z}_{\boldsymbol{m}}^{(k)} \quad For\ m = 1,2,\ldots,n-1 \tag{16}$$

- **Step 3**
- Update the LF state vector using (17).

$$\boldsymbol{x}^{(k+1)} = {}^{1}/_{2}\left[\boldsymbol{z}_{\boldsymbol{n}}^{(k)} + \boldsymbol{z}_{\boldsymbol{n-1}}^{(k)} + h\Delta\boldsymbol{z}_{\boldsymbol{n}}^{(k)}\right] \tag{17}$$

- **Step 4.**
- Set $k = k + 1$ and calculate the mismatches vector $\mathbf{g}^{(k)}$
- **Step 5.**
- Check the convergence criteria (6). If the convergence is achieved or the maximum number of iterations is surpassed, stop, otherwise go to step 6.
- **Step 6.**
- Update the "big" step size using the following rule:

$$\begin{aligned} &\zeta = \left\|\boldsymbol{z}_{\boldsymbol{n}}^{(k)} - \boldsymbol{z}_{\boldsymbol{n-1}}^{(k)}\right\|_{\infty} \\ &if\ \zeta > \varepsilon\ then\ H \leftarrow max\{\sigma_1 H, H_{MIN}\} \\ &if\ \zeta \leq \varepsilon\ then\ H \leftarrow min\{\sigma_2 H, H_{MAX}\} \end{aligned} \tag{18}$$

where ε is the security factor, σ_1 and σ_2 are damping coefficients and H_{MIN} and H_{MAX} are the minimum and maximum step size, respectively. Despite this rule is simple but effective and reliable. It uses the index ζ to measure the algorithm progress, thus, it is able to conveniently reduce and increase the step size (see Section V for more details about expression (18)).

- Update the value of n following the series (11) or (12) and calculate h using (10). If $n = n_{MAX}$, stop the updating.
- Calculate the LF Jacobi matrix $\mathbf{g}_{\mathbf{x}}^{(\mathrm{k})}$.
- Go to step 2.

The above steps can be summarized in Fig. 2.

Being faithful to the standard BS procedure, the Richardson Extrapolation should be recurrently used after the modified midpoint rule to refine the value of the LF state vector. Nevertheless, this step has been omitted in the proposed LF solver. We have observed that good results can be achieved without including this step, therefore, it is recommendable to remove it in order to reduce the computational effort.

It is important to notice that each iteration of the BS method in LF problem computes up to n inversions of the LF Jacobi matrix. Hence, it is important to balance the requirements of robustness and computational efficiency in order to choose a good value to the truncation. In Section V, we provide a guideline to properly choose the values of n_{MAX} and n_{MIN}.

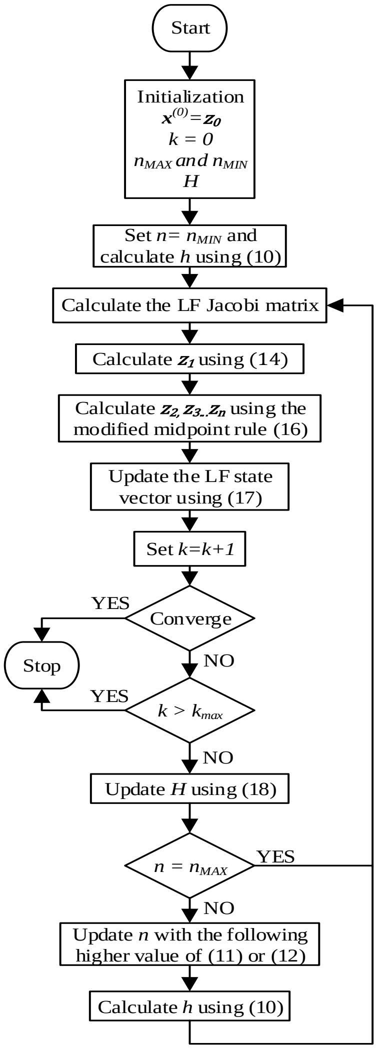

Fig. 2 LF solution process based on the proposed BS algorithm.

IV. The Reverse-Bulirsch-Stoer

The BS was originally conceived for set of ODEs, hence, in this case it is convenient to start with the smallest value of n and increase it until the desirable accuracy is reached. Thus, expensive steps are not computed if it is not necessary. However, in LF analysis, it would be convenient starting the iterative procedure with $n = n_{MAX}$, then, n would be reduced until n_{MIN} (e.g. $n = 2$). This is justified the concept of an iterative algorithm: when an iterative procedure is susceptible to diverge, the most difficult part is the computation of the first steps. Hence, the main idea would be using the most exact steps (i.e. the biggest n) at first iterations and then would be convenient to change to bigger steps in order to accelerate the convergence. This philosophy, applied to the BS, has blossomed in the proposed Reverse-Bulirsch-Stoer method (RBS). For the sake of exemplary, Fig. 3 shows the procedure of the BS and the RBS, for a truncation at $n = 6$, are compared. From this figure, it can be observed that the BS tends to increase the value of n, while the behaviour of the RBS is just the opposite.

Hence, the main idea behind the RBS would be to reduce the number of matrix inversions computed while the robustness is preserved. For example, let us consider the case of BS, we start with $n = 2$ and it would be increased up to n_{MAX}. Then, iterations are computed repeatedly with $n = 6$ until the convergence is reached (if the number of iterations is higher than 3), therefore, the BS needs to compute up to $6 + 6(K - 2)$ (being K the total number of iterations) number of matrix inversions. On the other hand, let us consider the same case for RBS, we would start with $n = 6$ and reduce it until $n = 2$, then, iterations would be computed repeatedly with $n = 2$ instead of $n = 6$ as it was the case for the BS. Thus, the RBS needs to compute $10 + 2(K - 2)$ matrix inversions. Considering the reasonable value $K = 10$, the BS computes up to 54 matrix inversions, 28 more than the RBS.

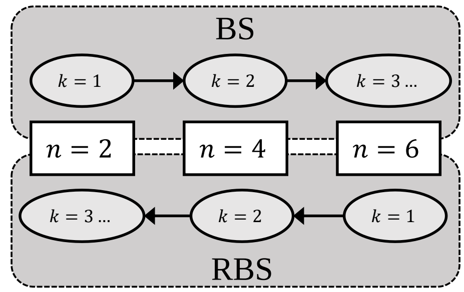

Fig. 3 A comparison between BS and RBS.

The proposed flowchart of the Fig. 2 is also valid for the proposed RBS. However, in the case of the RBS, $n = n_{MAX}$ is initially taken. Then, it is reduced each iteration until n_{MIN}.

V. Practical Recommendations

The proposed BS and RBS need to be initialized by defining a set of parameters. In this subsection, we try to facility a guideline to properly choose them, basing on our personal experience and heuristic criteria.

The series (11) and (12) need to be upper and lower bounds by defining n_{MAX} and n_{MIN}, respectively. Both values have a direct impact in the performance of the proposed methods. If they are chosen too high, the efficiency of the method is affected, however, it has to be expected higher robustness. Therefore, they should be selected as small as it is possible. Obviously, it is mandatory that $n_{MAX} \geq n_{MIN}$.

The "big" step size is adapted each iteration according to (18), nevertheless, it has to be initialized. The initial H has a significant impact on the performance of the proposed methods. It should be chosen as big as it is possible, however, the algorithm trends to diverge if H is initially taken too high. Generally, the RBS allows a higher initial step size than BS.

Several parameters have to be defined to adapt the step size according to (18). It is difficult to know a priori the good value of ζ, it mainly depends on the studied case. It should be chosen in the manner which the step size reduced at first iterations and progressively increased it as the algorithm progresses and the convergence is ensured. Regarding to σ_1 and σ_2, they should be near of 1. Since σ_1 is destined to reduce the step size and σ_2 to increase it, it should be satisfied that $\sigma_1 < 1$ and $\sigma_2 \geq 1$. The step size can be upper and lower bounds by defining H_{MAX} and H_{MIN}, respectively. There is no a specific rule to define them, but they should not be much different with respect to the initial step size taken.

The values of different parameters that have been considered for the simulations are given in Table I. Moreover, different recommended ranges for these parameters are facilitated.

TABLE I
VALUES OF THE DIFFERENT PARAMETERS AND RECOMMENDED RANGES

Parameter	Employed	Recommended
$\boldsymbol{n_{MAX}}$	6	(6, 12]
$\boldsymbol{n_{MIN}}$	2	(2, 4]
Initial $\boldsymbol{H}$	0.75 (BS) 1 (RBS)	(0.5, 0.75] (BS) (0.75, 1] (RBS)
$\boldsymbol{\zeta}$	8	(3, 8]
$\boldsymbol{\sigma_1}$	0.95	(0.85, 0.95]
$\boldsymbol{\sigma_2}$	1.05	[1.05, 1.10]
$\boldsymbol{H_{MAX}}$	2	[1.2, 2]
$\boldsymbol{H_{MIN}}$	0.5	[0.5, 0.7]

Parameters employed in simulations are guidelines and one can conclude that they work quite well in the studied cases. Nevertheless, each user can adapt them freely. For instance, the value of n_{MAX} and n_{MIN} may be as higher as necessary for difficult cases, oppositely, it can be reduced in well-conditioned scenarios. Even, in an extreme case, n_{MAX} can be tuned equal to n_{MIN}, so, the value of n would be fixed and conveniently high or small. Similar idea can be extrapolated for the initial value of H. As can be seen, the proposed methods are very versatile due to they can be adapted for convenience. We have also provided recommended intervals with the aim to provide a guideline as general as possible.

VI. Tests and Results

The proposed methods have been tested using the following very large-scale systems:

- The 3012-bus 3572-line model of the Polish system in the winter 2007-08 evening peak.
- The 3375-bus 4161-line model of the Polish system in the winter 2007-08 evening peak.
- The 13659-bus 20467-line portion of the European transmission system (further details of this system can be found in [31, 32]).

These systems are available to download in [33]. Furthermore, the proposed LF methods have been compared with the following LF methods:

- Standard NR method.
- Hybrid Gauss-Newton method: in this method, the iterative procedure starts with the Gauss-Seidel method [34], and switch to the standard NR when $\|\mathbf{g}\|_\infty \leq 10^{-2}$
- Iwamoto's method [9].
- Forward-Euler method [5].
- 4th order Runge-Kutta formula [5].
- Levenberg-Marquardt method [16].
- High-order Levenberg-Marquardt method [17].
- Non-monotone line search with corrected Levenberg–Marquardt method [18].
- Lyapunov method [16].
- Dynamic LF approach [19, 20].

The proposed and other LF methods have been programmed by modifying the original code of MATPOWER 6.0 [35]. MATPOWER is an open source platform and widely used in several references. Therefore, its standard code can be freely modified in order to incorporate other LF methods. In order to check the developed codes of these methods, several small and medium-scale well-conditioned systems have been solved, and the obtained results have been compared with those obtained by standard NR which is already implemented in MATPOWER software. We have used some of test systems provided by MATPOWER (e.g. IEEE test cases, some cases of the PEGASE project…). As well-known that the LF solution of such well-conditioned systems using any valid algorithm must be identical with the standard NR method with different features such as number of iterations and computation time, hence this step is used to check the correct programming of the algorithm.

The performance of the proposed methods has been validated when generator's reactive power limits are violated. In the course of LF solution, the calculated reactive power of each generator is checked if it is within the limits or not. If there is a violation, then it is set equal to the limit violated and the bus voltage magnitude is released, where the connected bus is converted to PQ bus [5, 35, 36]. The LF Jacobi matrix must be consequently modified in order to incorporate the reactive power mismatches (2). Then, the LF is repeatedly solved until a feasible solution is achieved.

Table II shows a comparison of the performance of different LF solvers in the studied cases using a flat initial guess and $\epsilon = 10^{-4}$. All simulations have been carried out using a Personal Computer with a 3.4 GHz Intel Core i5-7500 CPU.

Results considering generator's reactive power limits are given in Table III (using a flat initial guess). It can be observed that the proposed methods are still effective when generator's reactive power limits are considered.

Fig. 4, 5 and 6 show the convergence characteristics of the proposed methods for the studied cases.

TABLE II
COMPARISON OF VARIOUS LF METHODS FOR THE STUDIED CASES

Method		3012-bus		3375-bus		13659-bus	
		Time [s]	# iter	Time [s]	# iter	Time [s]	# iter
Standard NR		Diverge	--	Diverge	--	Diverge	--
Gauss-Newton		Diverge	--	Diverge	--	Diverge	--
Iwamoto [9]		Fail*	--	Fail*	--	Fail***	--
Explicit Euler [5]		Diverge	--	Diverge	--	Fail***	--
4th order Runge-Kutta [5]		Diverge	--	Diverge	--	6.30	18
Levenberg methods	**[16]**	Fail*	--	Fail*	--	Fail*	--
	[17]	Fail*	--	Fail*	--	Fail**	--
	[18]	Fail*	--	Fail*	--	Fail**	--
Lyapunov method	**[16]**	Fail*	--	Fail*	--	Fail*	--
Dynamic LF approach	**[19, 20]**	Fail**	--	Fail**	--	Fail**	--
BS		1.77	15	2.12	16	8.65	15
RBS		0.79	12	0.86	12	3.75	12

** It did not converge in 50 iterations*
*** It did not get results in a reasonable time*
**** It provided a wrong solution*

TABLE III
COMPARISON OF VARIOUS LF METHODS FOR THE STUDIED CASES CONSIDERING GENERATOR'S REACTIVE POWER LIMITS

Method		3012-bus		3375-bus		13659-bus	
		Time [s]	# iter	Time [s]	# iter	Time [s]	# iter
Standard NR		Diverge	--	Diverge	--	Diverge	--
Gauss-Newton		Diverge	--	Diverge	--	Diverge	--
Iwamoto [9]		Fail*	--	Fail*	--	Fail***	--
Explicit Euler [5]		Diverge	--	Diverge	--	Fail***	--
4th order Runge-Kutta [5]		Diverge	--	Diverge	--	9.86	29
Levenberg methods	**[16]**	Fail*	--	Fail*	--	Fail*	--
	[17]	Fail*	--	Fail*	--	Fail**	--
	[18]	Fail*	--	Fail*	--	Fail**	--
Lyapunov method	**[16]**	Fail*	--	Fail*	--	Fail*	--
Dynamic LF approach	**[19, 20]**	Fail**	--	Fail**	--	Fail**	--
BS		3.56	31	4.15	33	14.06	25
RBS		1.65	24	2.03	25	6.29	20

** It did not converge in 50 iterations*
*** It did not get results in a reasonable time*
**** It provided a wrong solution*

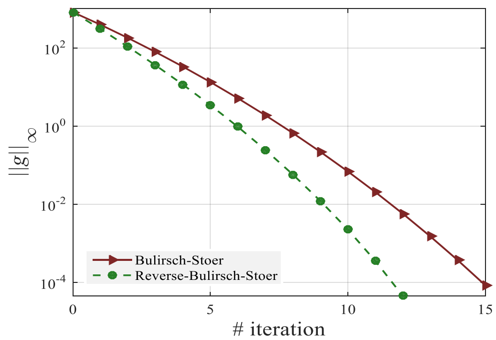

Fig. 4 Convergence characteristics of the proposed methods (3012-bus system).

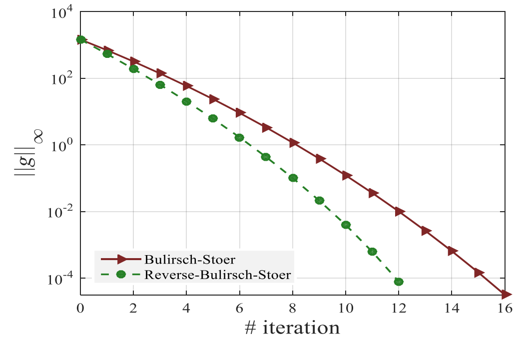

Fig. 5 Convergence characteristics of the proposed methods (3375-bus system)

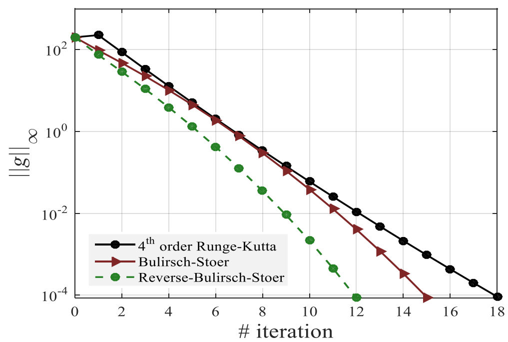

Fig. 6 Convergence characteristics of the proposed methods (13659-bussystem)

The proposed methods have converged efficiently in all cases, while the remainder methods have mainly failed due to the lack of a good initial guess point, which has not been close enough or far away the correct solution, consequently the algorithms become too slow or unstable.

In the 13659-bus case, the 4th order Runge-Kutta has converged in 18 iterations, while the proposed BS and RBS have employed 15 and 12 iterations, respectively. The computation time of the BS has been higher than the 4th order Runge-Kutta. It has expected due to after iteration no. 3 the BS needs 6 matrix inversions per iteration. This drawback is overcome by the proposed RBS, which improves the performance of the 4th order Runge-Kutta and the proposed BS in both computation time and number of iterations.

The proposed RBS ensures a higher efficiency than the BS, since the RBS needs less matrix inversions than the BS. Also, the RBS always provides a smaller value of $\|\mathbf{g}\|_\infty$ than the BS.

The “big” step size has been adapted during the iterative procedure according to (18). The main idea behind this adaptive mechanism is to reduce the step size only if it is strictly necessary in order to avoid the failure of the algorithm. Thus, it has to be expected that the step size grows during the iterative progress in normal circumstances. Fig. 7 shows the evolution of the step size in the studied cases. The step size is only reduced at the first iteration in the 13659-bus case. In the remainder cases, the step size is progressively increased during the iterative process.

- Scalability of the proposed methods

Fig. 8 shows the computation time employed for the proposed BS and RBS in the different studied cases. As can be easily noticed, the BS’s trend line (dotted red line) is steeper than the RBS’s trend line (dotted green line). It means that the proposed BS is more sensitive with respect the size of the system than the RBS. This is due to the BS needs to compute many more matrix inversions during the iterative process (See Fig. 9). Basing on these ideas, one can conclude that the RBS is more scalable than the BS.

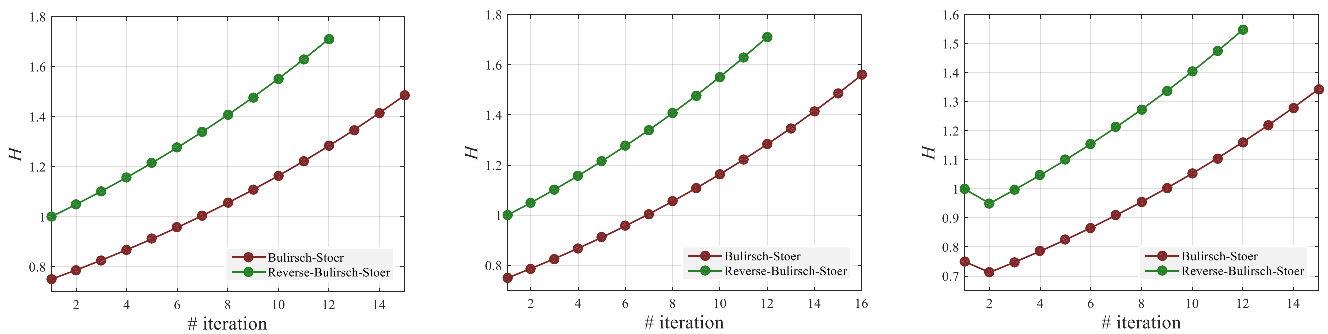

Fig. 7 Step size during the iterative process in the 3012-bus (left), 3375-bus (middle) and 13659-bus (right) cases.

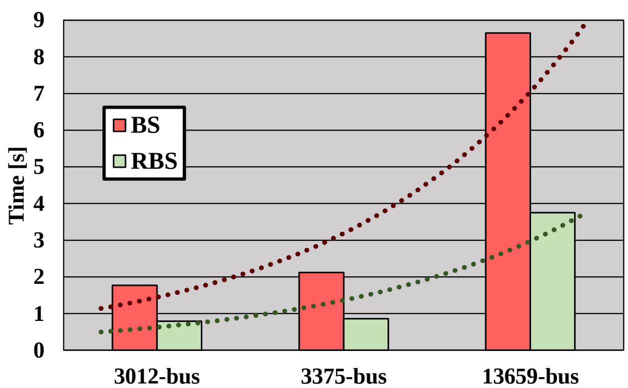

Fig. 8 Computation time employed for the proposed methods in the studied cases.

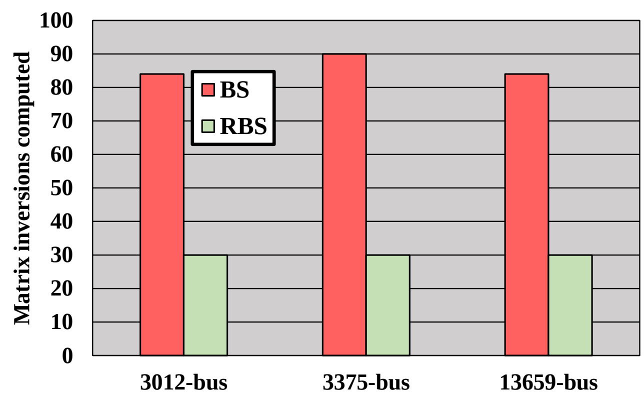

Fig. 9 Required matrix inversions during the iterative process for the studied cases.

- Performance of the proposed methods near to the maximum loadability point

It is well-known that the condition of the system becomes ill when the loading level is near to the maximum loadability point. The ability of the proposed methods for solving the LF problem of the studied cases when they are operation at this situation has been explored. Hence, the loading profile of the system is modified as follows:

$$P_i^{sch} = \rho P_i^{sch} \quad \forall i \in [\boldsymbol{PQ} \cup \boldsymbol{PV}] \qquad (19)$$
$$Q_i^{sch} = \rho Q_i^{sch} \quad \forall i \in [\boldsymbol{PQ}] \qquad (20)$$

where, ρ is a real positive coefficient which represents the loading level and $\boldsymbol{PV}$ and $\boldsymbol{PQ}$ are the vectors of PV and PQ buses indices respectively. Table IV gives the total number of iterations of the proposed methods for the 3012-bus and 3375-bus cases at different loading conditions using a flat initial guess. We have not included results for the 13659-bus since the base loading level of this case is practically its maximum loadability condition.

TABLE IV
TOTAL NUMBER OF ITERATIONS EMPLOYED BY THE PROPOSED LF METHODS AT DIFFERENT LOADING LEVELS

ρ	3012-bus		ρ	3375-bus	
	BS	**RBS**		**BS**	**RBS**
1.05	15	12	**1.05**	16	12
1.10	15	12	**1.07**	16	12
1.15	15	12	**1.10**	16	12
1.20	15	12	**1.12**	16	12
1.27	15	12	**1.15**	16	12
1.28	Diverge	Diverge	**1.16**	Diverge	Diverge

The proposed methods are able to converge with different loading conditions employing the same number of iterations in all cases. Therefore, the ability of the proposed methods for efficiently solving the LF even near to the maximum loadability point has been proven.

- Performance of the proposed methods with respect to initial guess points

In all previous simulations, the flat initial guess has been considered for solving LF problem. Now let us explore the performance of the proposed methods using poorer initial guess points. To do that, the flat initial guess has been deteriorated by introducing a real factor e. Hence, the initial guess can be calculated as follows:

$$\boldsymbol{V}_{\boldsymbol{PQ}}^{(\boldsymbol{0})} = 1.0 + e \qquad (23)$$

where, $\boldsymbol{V}_{\boldsymbol{PQ}}^{(0)}$ is the vector of initial voltage magnitudes at PQ buses. The comparison has been achieved for different values of e. In Table V, the results obtained by the proposed LF methods have been compared with those obtained by 4th order Runge-Kutta [5].

From Table V, it can be observed that the proposed methods show outstanding robustness with respect to the initial guess. The 13659-bus system is the hardest case, as can be seen, the factor e is smaller than remainder cases. Anyway, the 4th order Runge-Kutta starts to fail with $e = 0.005$ while the proposed methods are still able to successfully converge.

TABLE V
TOTAL NUMBER OF ITERATIONS OF THE STUDIED CASES FOR DIFFERENT INITIAL GUESSES

Real factor e	**4th order Runge-Kutta [5]**	**BS**	**RBS**
	3012-bus case		
$\mathbf{0.05}$	Diverge	15	12
$\mathbf{0.1}$	Diverge	16	12
$\mathbf{0.15}$	Diverge	16	13
	3375-bus case		
$\mathbf{0.05}$	Diverge	16	12
$\mathbf{0.1}$	Diverge	16	12
$\mathbf{0.15}$	Diverge	16	13
	13659-bus case		
$\mathbf{0.001}$	17	15	12
$\mathbf{0.005}$	Fail***	16	13

**** It provided a wrong solution*

- Comparison among proposed methods and other Continuous Newton's methods

In order to demonstrate the superiority of the proposed techniques with respect other Continuous Newton's methods [5], let us assume that the step size is kept constant during the iterative procedure, thus, the robustness of the numerical scheme is tested without using exogenous factors like adaptive step size mechanism. Table VI presents the required number of iterations and computation time of different CN methods proposed in [5] and the proposed BS and RBS (a flat initial guess has been used for all cases).

TABLE VI
TOTAL NUMBER OF ITERATIONS AND COMPUTATION TIME OF VARIOUS CONTINUOUS NEWTON'S METHODS

Method	**3012-bus**		**3375-bus**		**13659-bus**	
	Time [s]	# iter	Time [s]	# iter	Time [s]	# iter
Explicit Euler [5]	Diverge	--	Diverge	--	Diverge	--
4th order Runge-Kutta [5]	Diverge	--	Diverge	--	5.67	16
BS	2.71	22	3.20	23	12.11	20
RBS	1.08	17	1.21	17	4.64	15

From this table, it can be observed that the other methods still fail to converge when a fixed step size strategy is used, while the proposed methods keep their robustness features. The bad performance of these methods is due to the step size tends to be increased during the iterative procedure in order to accelerate the convergence (see Fig 7) and in this case it is fixed and the algorithm tends to be slower. Therefore, using small step sizes is not a logical strategy when the method maintains its robustness features with bigger step sizes. Still so, the proposed

RBS is more efficient than the 4th order Runge-Kutta [5] (see results for the 13659-bus case).

The proposed BS and RBS provide a more sophisticated scheme which uses the modified midpoint rule with a non-constant number of intermediate points, therefore, it can be seen as a variable-order numerical scheme. To the best of our knowledge, this is a novel idea for solving the LF since the existing methods use the same order in all iterations (e.g. NR and Forward Euler are first order while the 4th order Runge-Kutta has order four).

VII. Conclusions

In this paper, two methods have been proposed to solve the LF problem of very large-scale systems. The proposed methods based on the Bulirsch-Stoer approach. The proposed methods have been validated using different large-scale systems (3012-bus, 3375-bus, and 13659-bus) and compared with several LF techniques; Standard NR, Hybrid approach Gauss-Newton, Iwamoto, Explicit Euler, 4th order Runge-Kutta, Levenberg methods, Lyapunov methods.

The obtained results have demonstrated the effectiveness of the proposed methods in solving the LF problem for very large-scale problems. They are able to find the LF solution efficiently in all studied cases, while the other methods showed different convergence difficulties:

- Divergence: in this case the value of $\|\mathbf{g}\|_\infty$ has grown during the iterative process. Therefore, the algorithm moves away to the solution. This failure is due to the flat initial guess has lied outside of the Region of Attraction.
- Too slow convergence: the algorithm has not converged in a specified number of iterations (considered 50 in the present paper). In this case, the flat initial guess may be inside of the Region of Attraction, but it is not close enough to the final solution.
- Wrong solution: the algorithm has converged in less of 50 iterations, but the solution is incorrect. In this case, the flat initial guess has lies inside of an incorrect Region of Attraction. Therefore, it has been "attracted" to a wrong solution (normally low voltage solution).
- Computational burden: it is normally produced by expensive computation which might be attributable to the size of the studied systems.

The aforementioned issues are mainly due to the flat initial guess, which occasionally has not been good enough, or low computational efficiency. The proposed methods are able to overcome these difficulties as they converged to a correct solution in less of 50 iterations avoiding computational problems in all studied cases.

However, the RBS is much less sensitive with respect to the size of the system than the BS due to the amount of matrix inversions computed during the iterative process. In the studied cases, the RBS has computed less matrix inversions than the BS. Hence, the RBS is expected to be more scalable to larger systems.

The ability of the proposed methods for efficiently solving the LF near to the maximum loadability point has been proven. Hence, the performance of the proposed BS and RBS in the Continuation Power Flow [36] is hopeful and it should be deeper explored in future works. The possibility to modify the values of the initial step size and n_{MAX} endows the proposed methods a great versatility. For example, we can use the more efficient schemes (i.e. big initial step size and small n_{MAX}) when the algorithm operated away from the maximum loadability point, and switch to safer parameters when the the maximum loadability point is approximated or a limit-induced bifurcation is reached.

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